Category Theory

An Introduction to Generic Abstractions

Wenqi Ding, 11/21/2021
Splash 2021: Math Without Numbers

I - Cats, Why, and Laziness of the Mind

How shall we describe a "thing"?

- Physical Manifestation?
- Psychological Effects?
- Symbolic Meaning?
- Applications?

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```
struct Apple {
  int particle_count;
  particle_t *particles;
  PAIR_T(particle_t *, particle_t *) bonds;

  mesh_t shape;
  albedo_t **uv_map;
};
```

How shall we describe a “thing”?

- Physical Manifestation
- Psychological Effects
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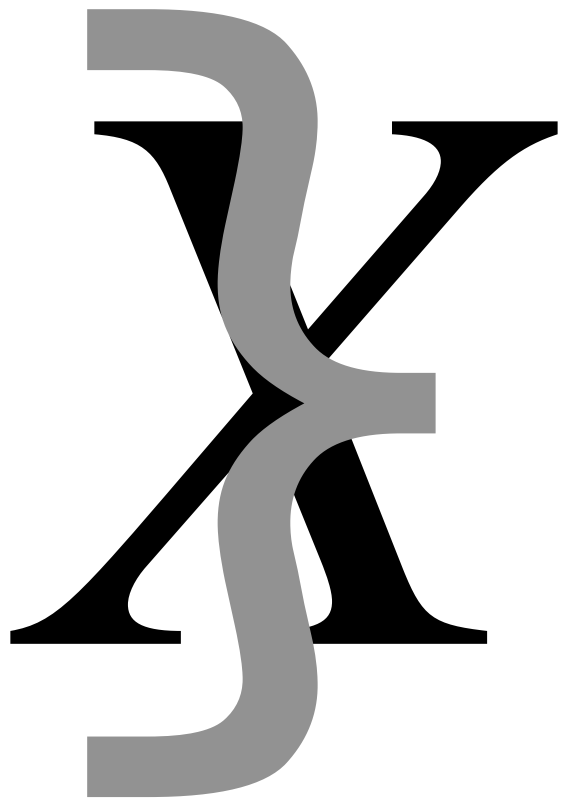
- Applications

```
struct Apple {
  MAP_T(culture_t *, effect_t *) meaning;

  int appeal;
  misc_t other_effects;
  // although descriptive, not very useful
  // for other fields
};
```

- Physical Manifestation
- Psychological Effects
- Symbolic Meaning
- Applications



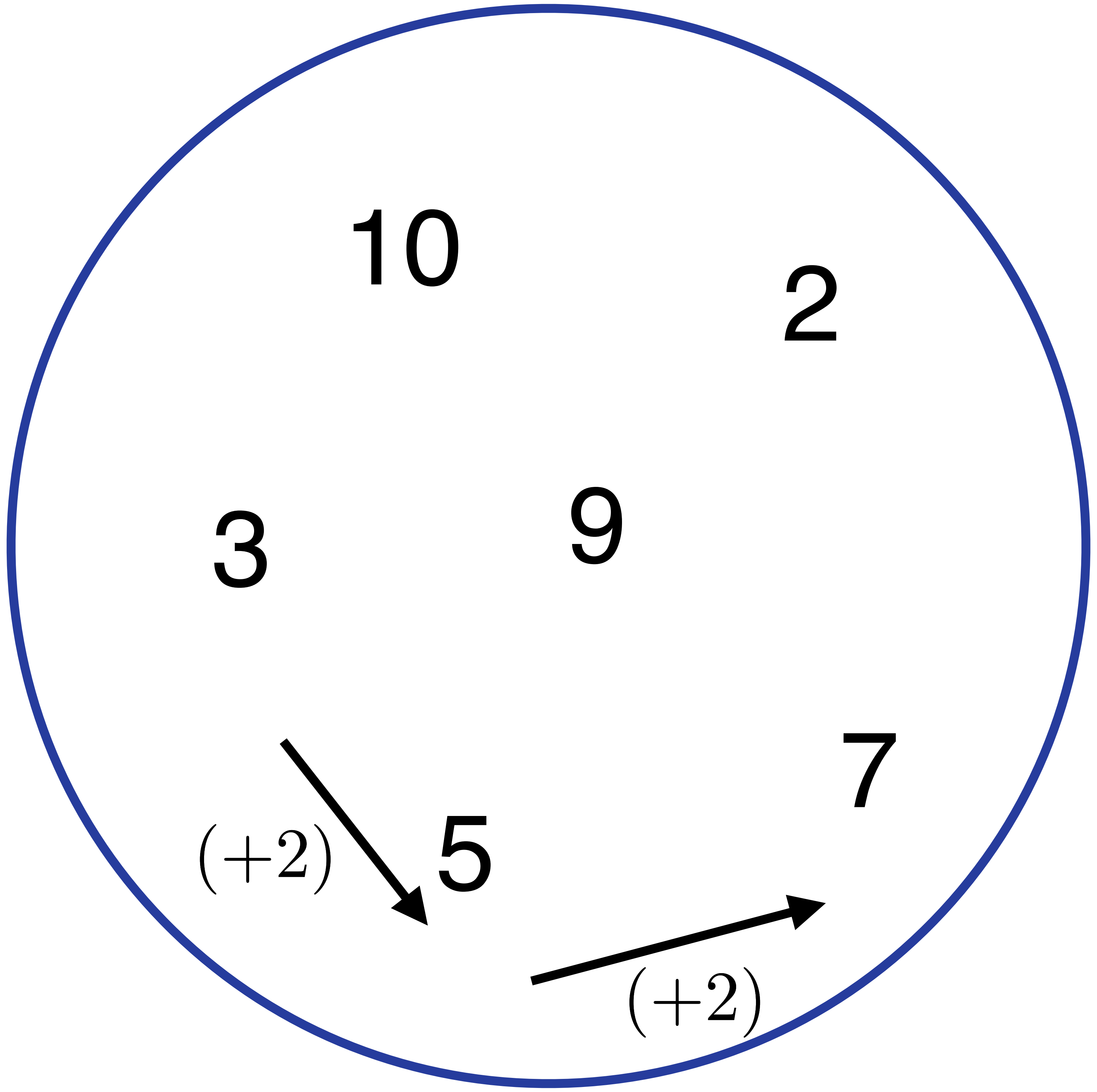
Independent of each other,
and takes up a lot of space

Category Theory: The study of objects in respect to their relationships with others*

*loosely speaking

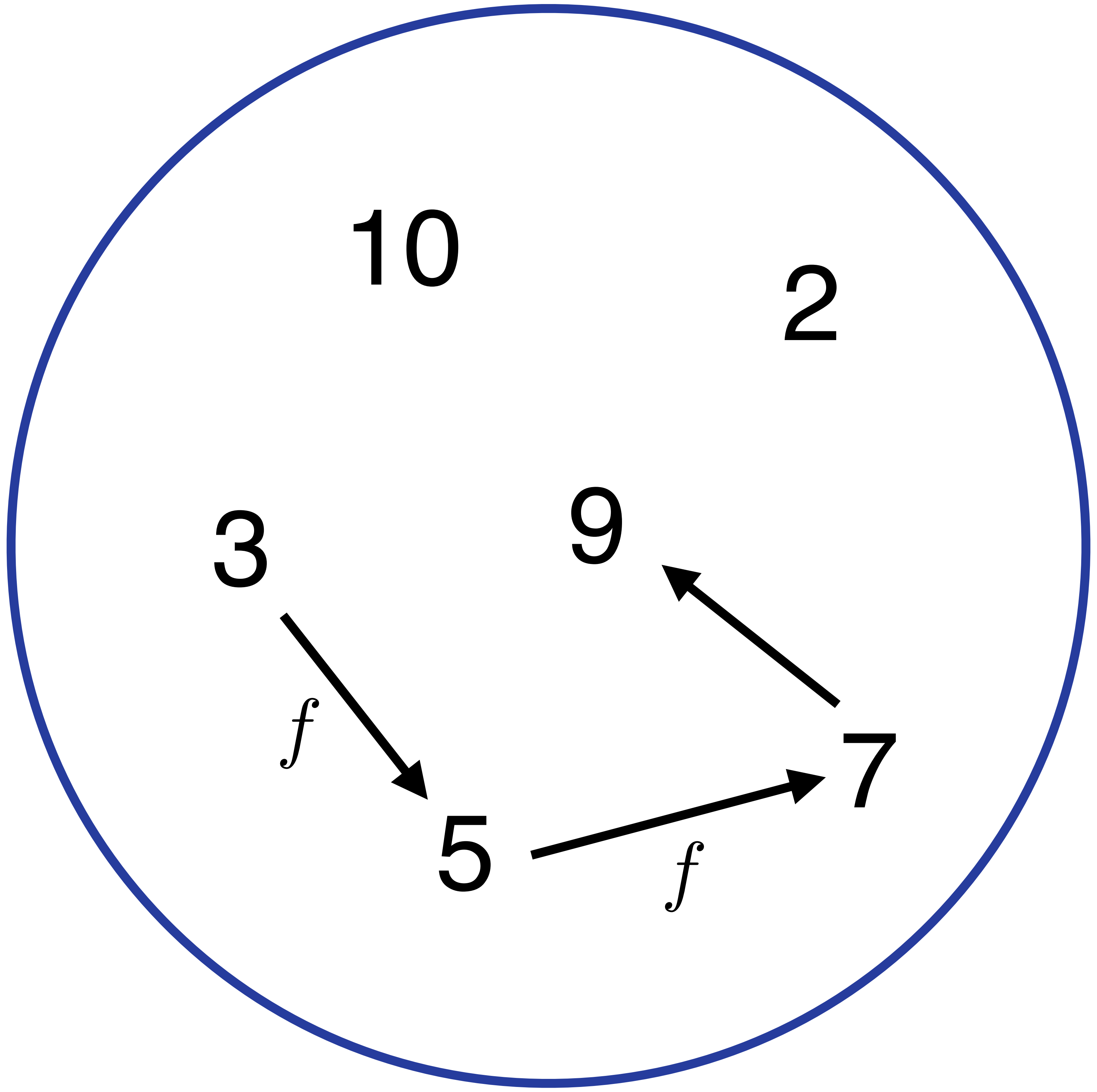
$$x \in \mathbb{N}$$

$(+2f) =$



$$x \in \mathbb{N}$$

$$f = (+2)$$



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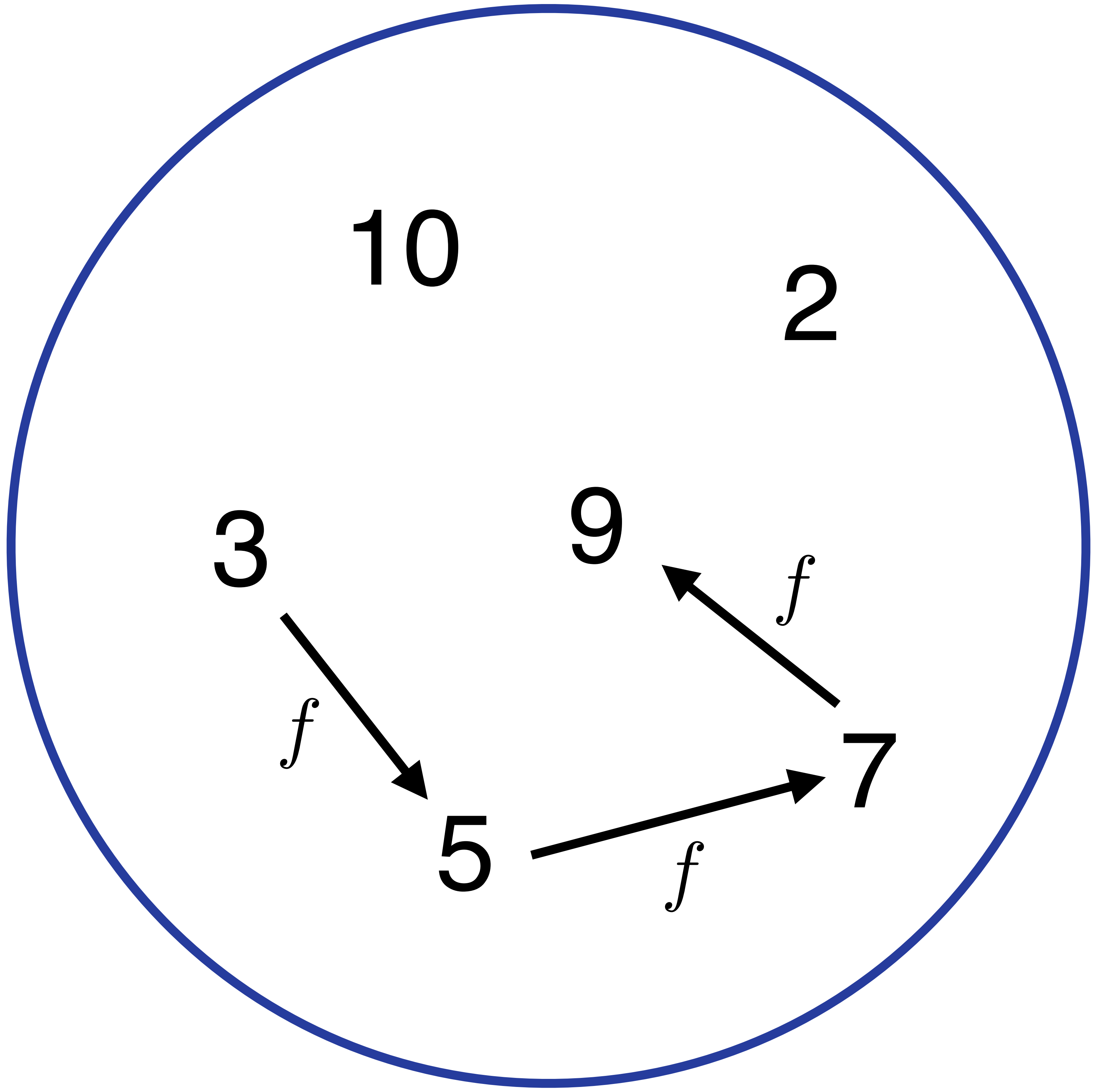

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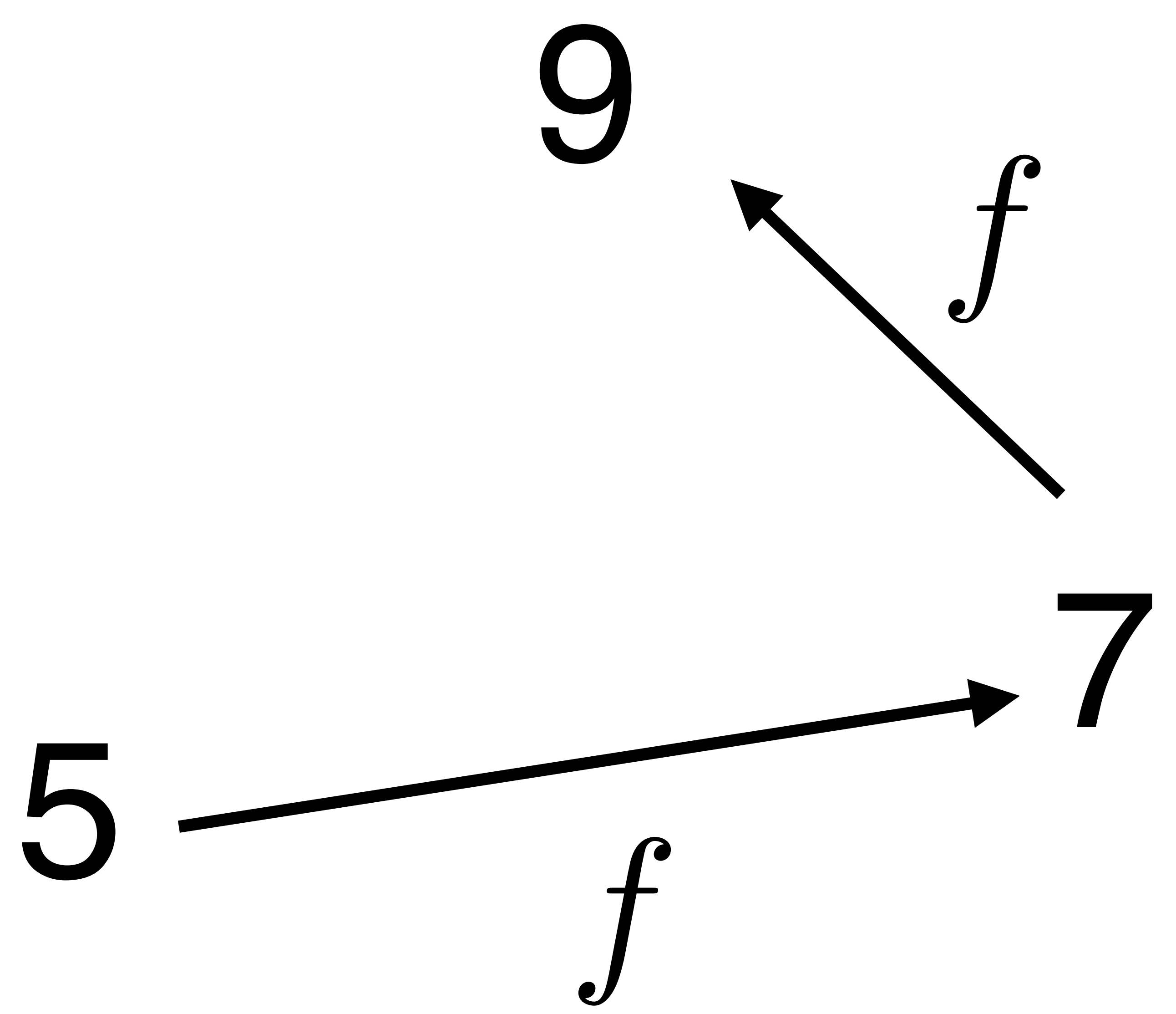
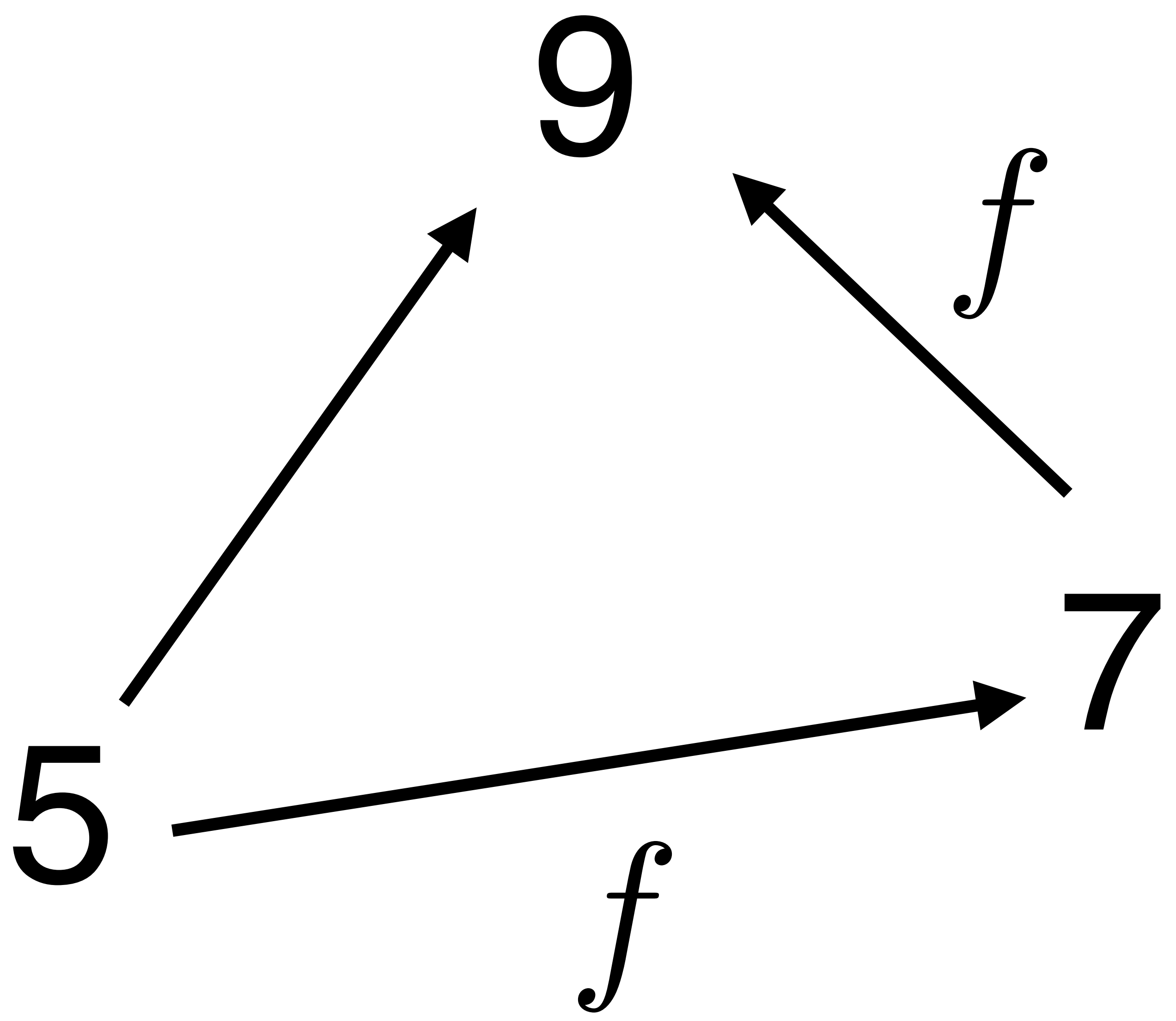


$$f = (+2)$$

$$f \circ f = (+2) \circ (+2)$$



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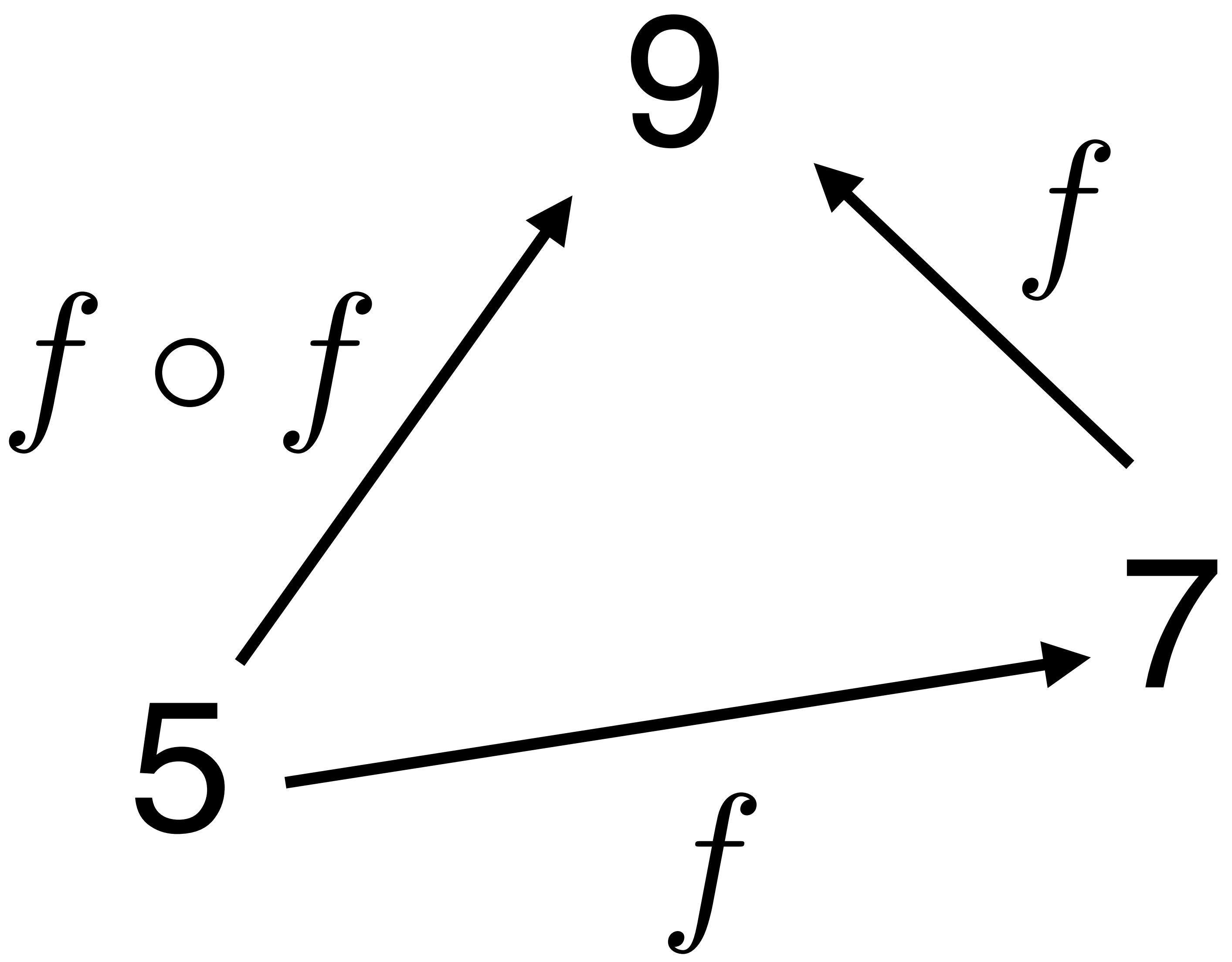
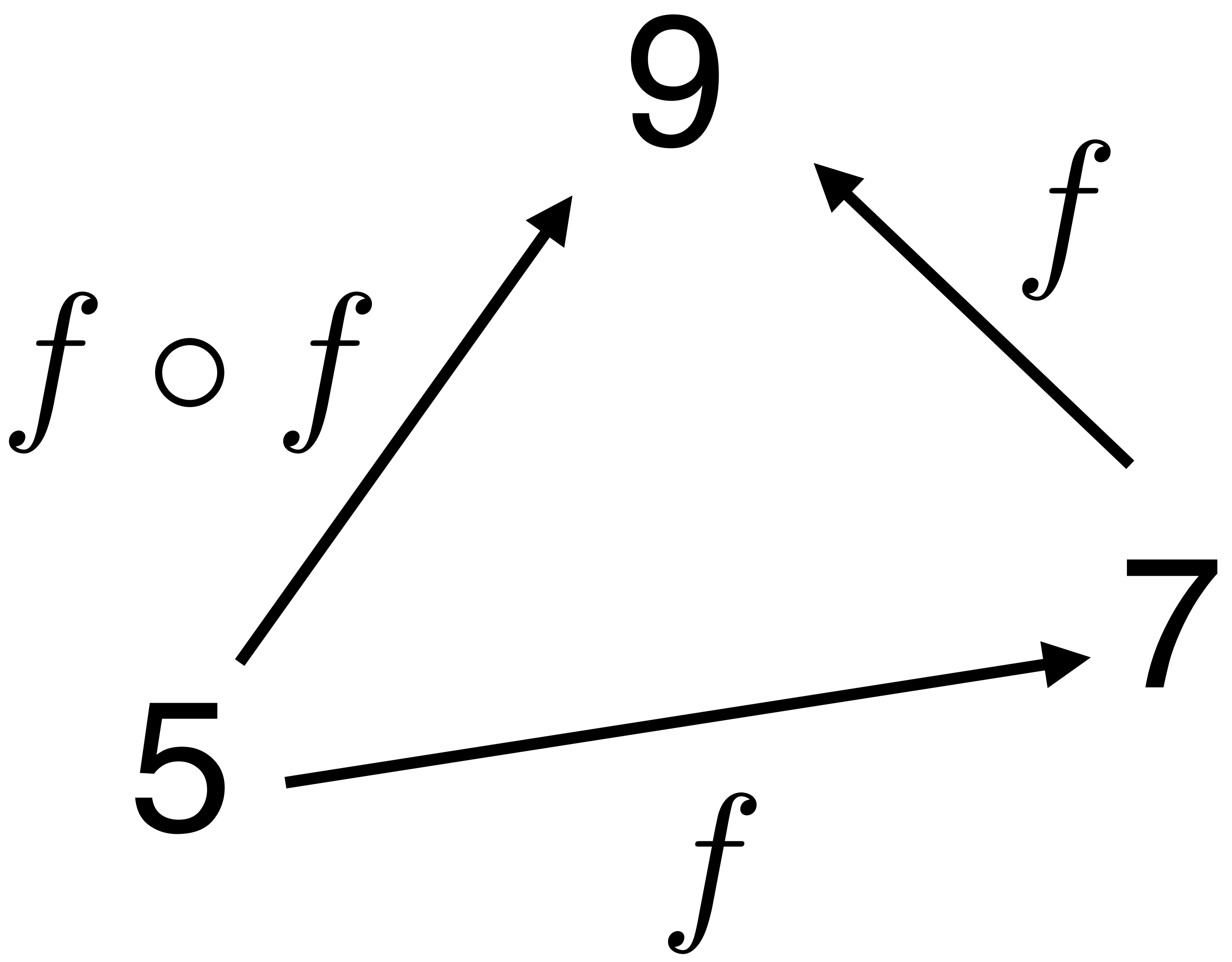
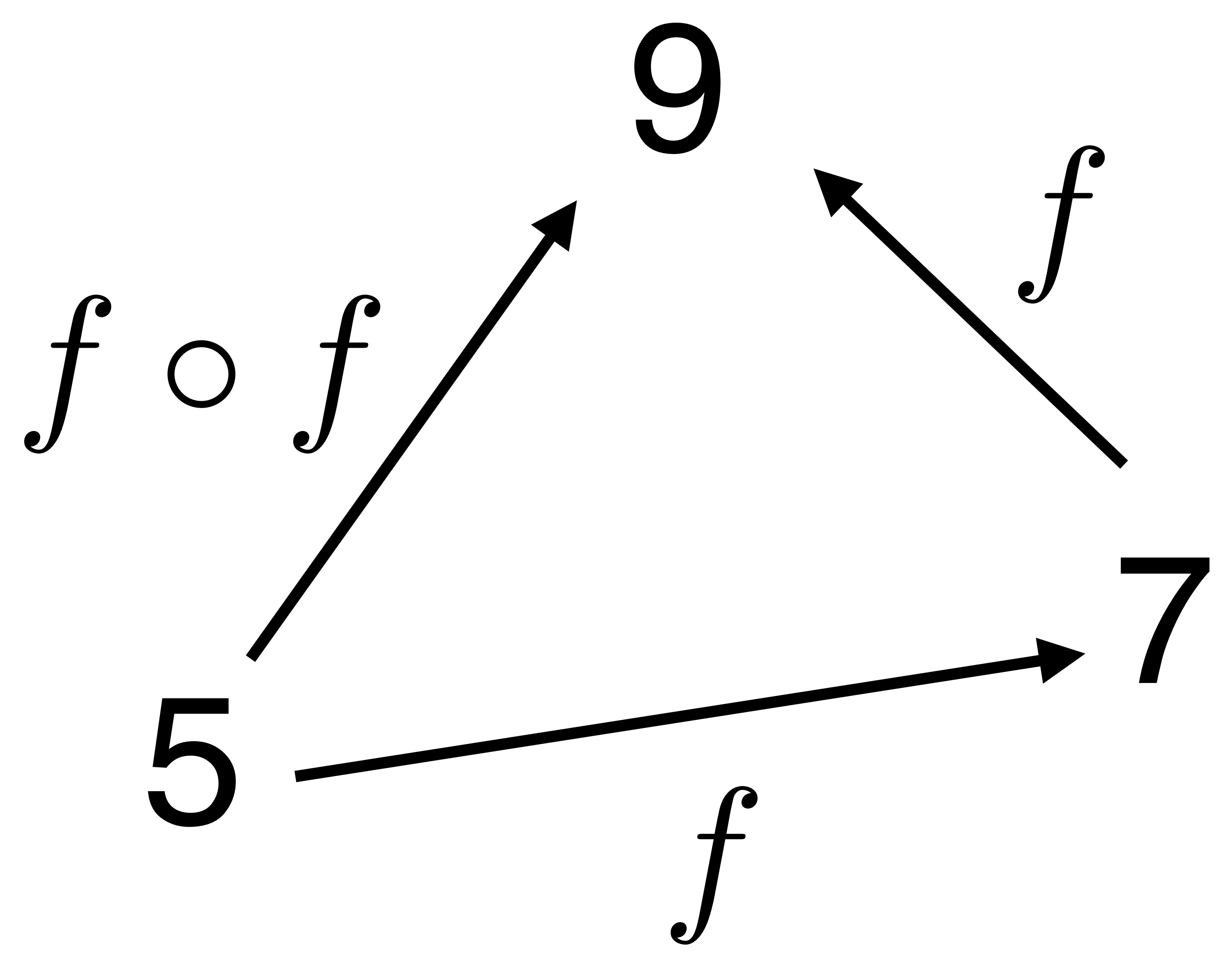


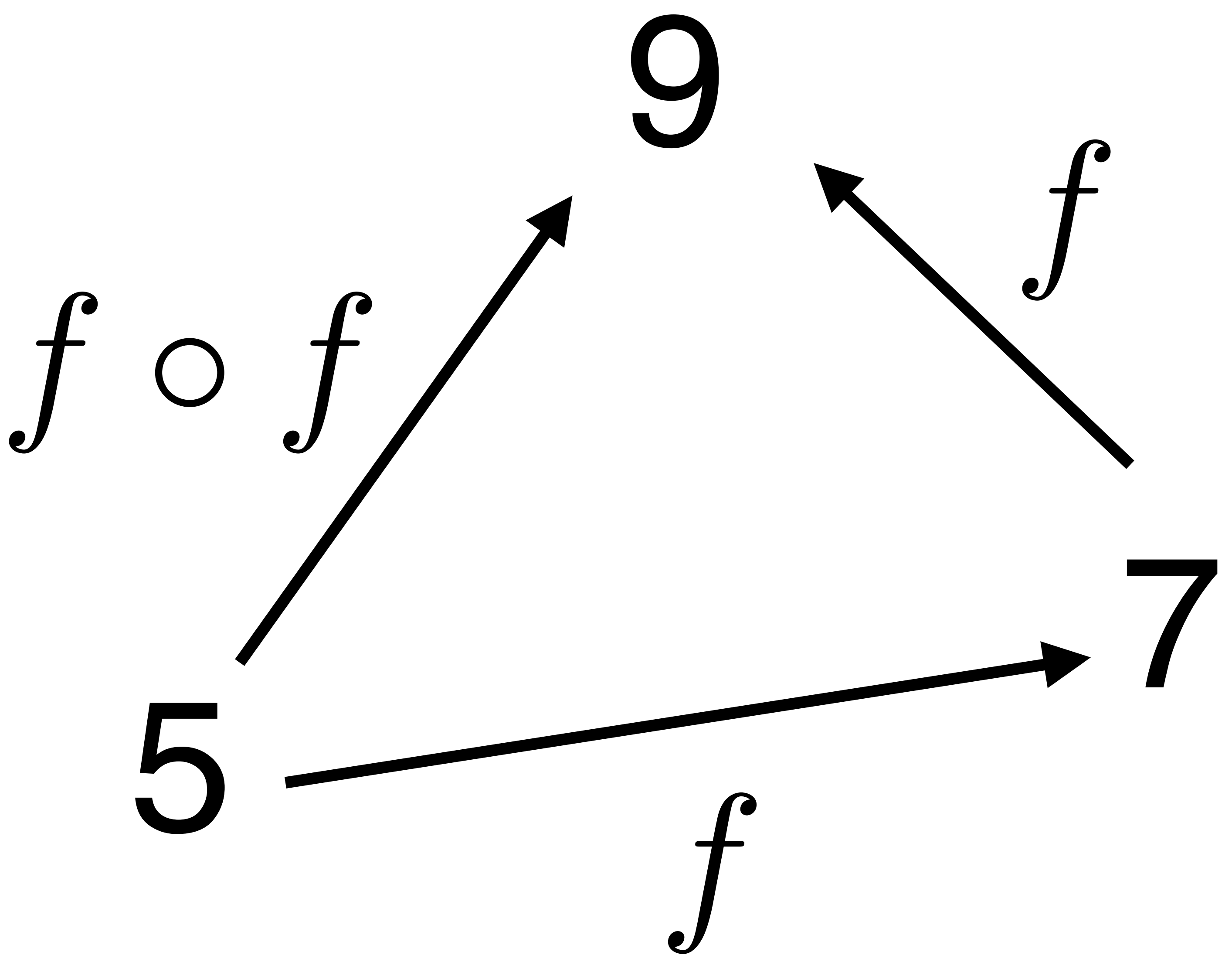
Arrows are Composable



Arrows are Transitive?



Arrows are Transitivity



Transitivity

Transitivity

Trans t ive ity

Trans t ive ity

across, through

Trans t ive ity

tending to…

Trans t ive ity

state or quality of

Trans t ive ity

Transitivity

Transitivity

Transitivity

The quality of being able to cross over

Transitivity

$$\forall a, b, c \in X : (aRb \wedge bRc) \Rightarrow aRc$$

Transitivity

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$$A \longrightarrow B \longrightarrow C$$

Transitivity

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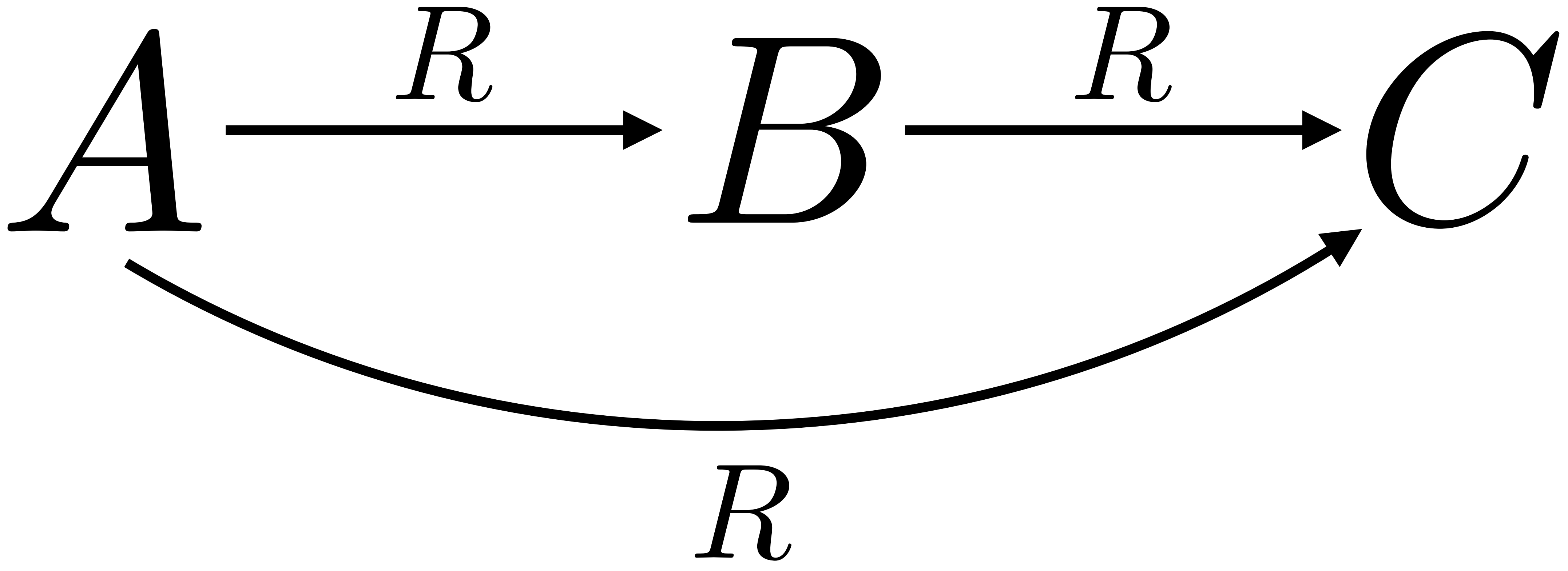

$$A \xrightarrow{R} B \xrightarrow{R} C$$

$$3 \xrightarrow{R} 5 \xrightarrow{R} 9$$

$$R = (<)$$

$$3 \longrightarrow 5 \longrightarrow 9$$

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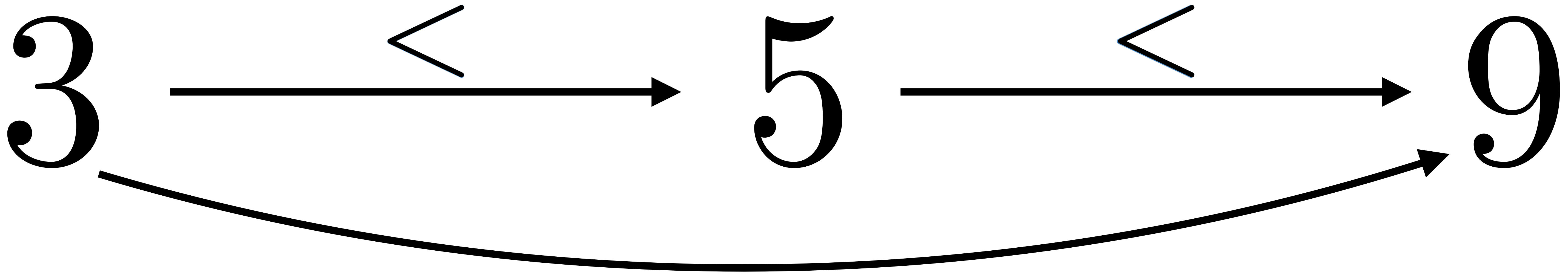
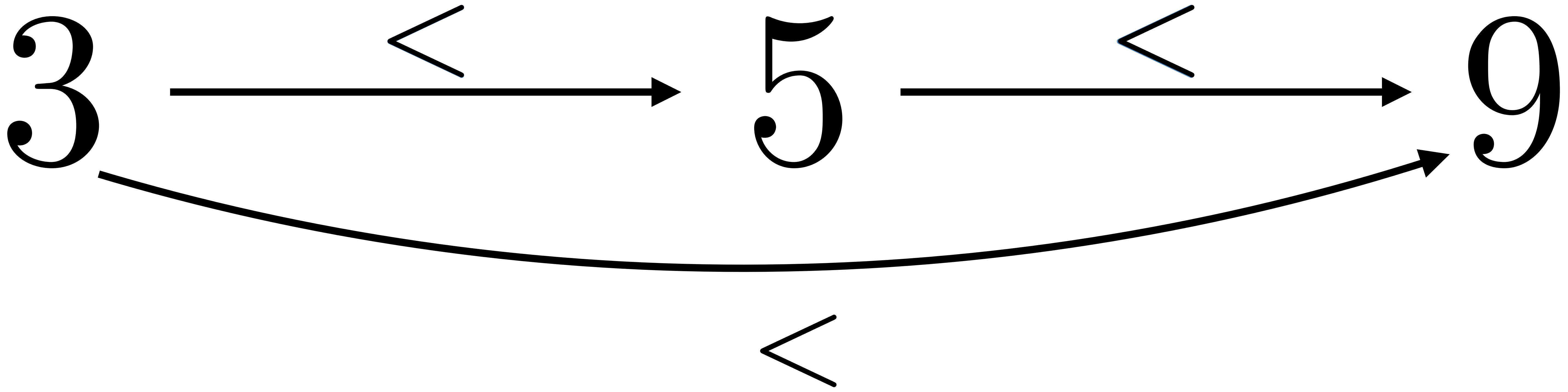
$$3 \xrightarrow{<} 5 \xrightarrow{<} 9$$

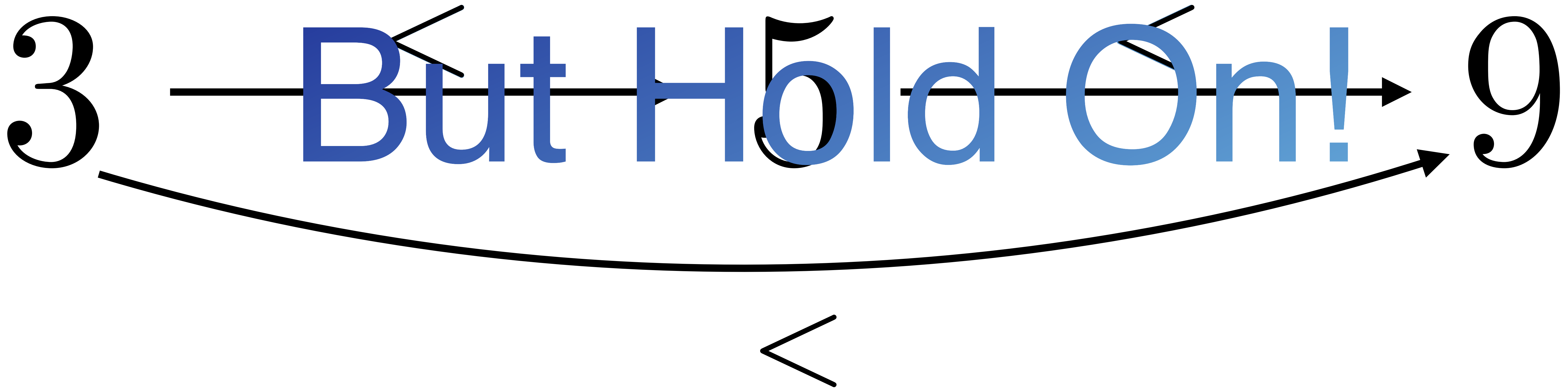
$$R = (<)$$



$$R = (<)$$



$$R = (<)$$



But Hold On!

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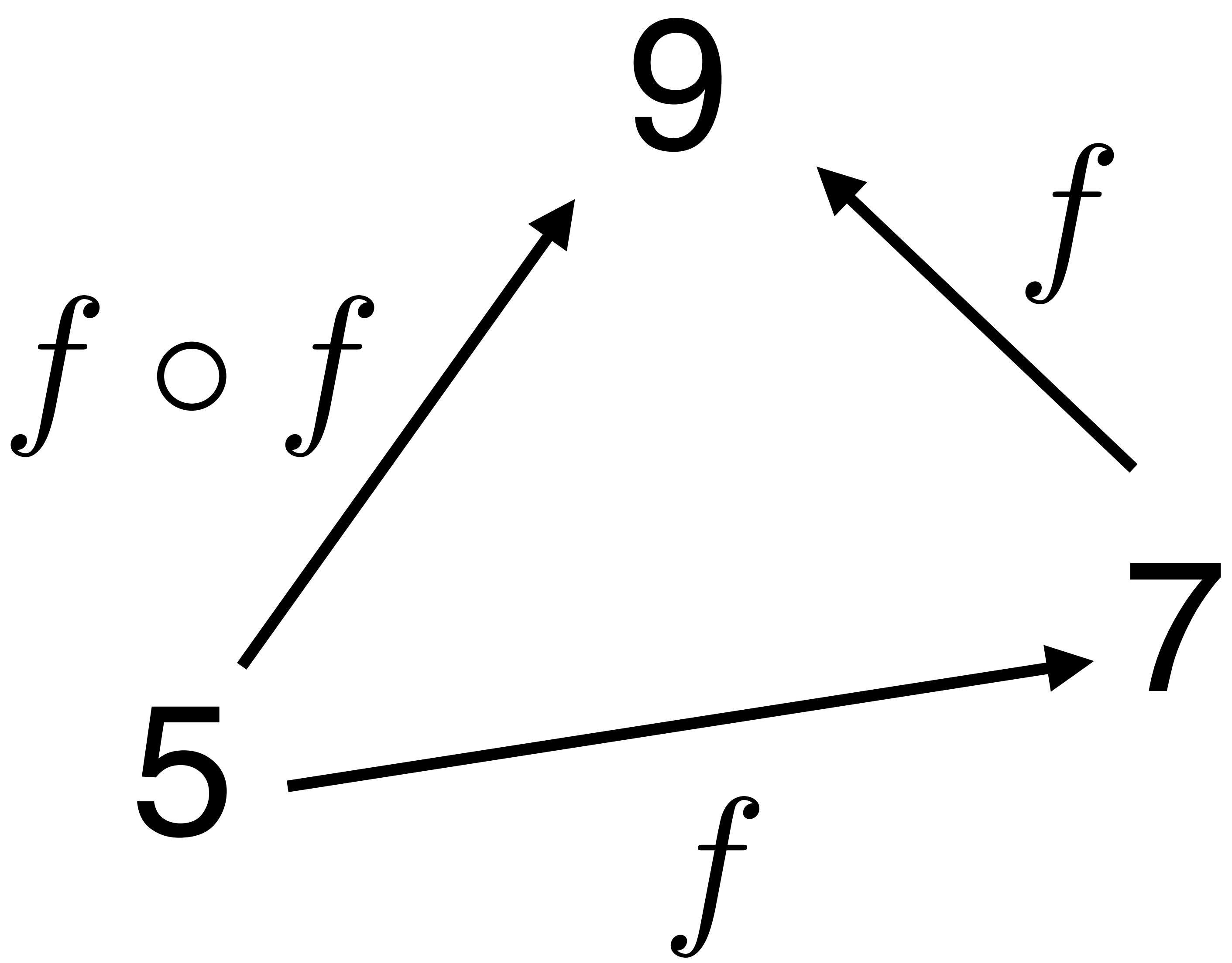
$$f = (+2)$$

$$f \circ f = (+2) \circ (+2)$$



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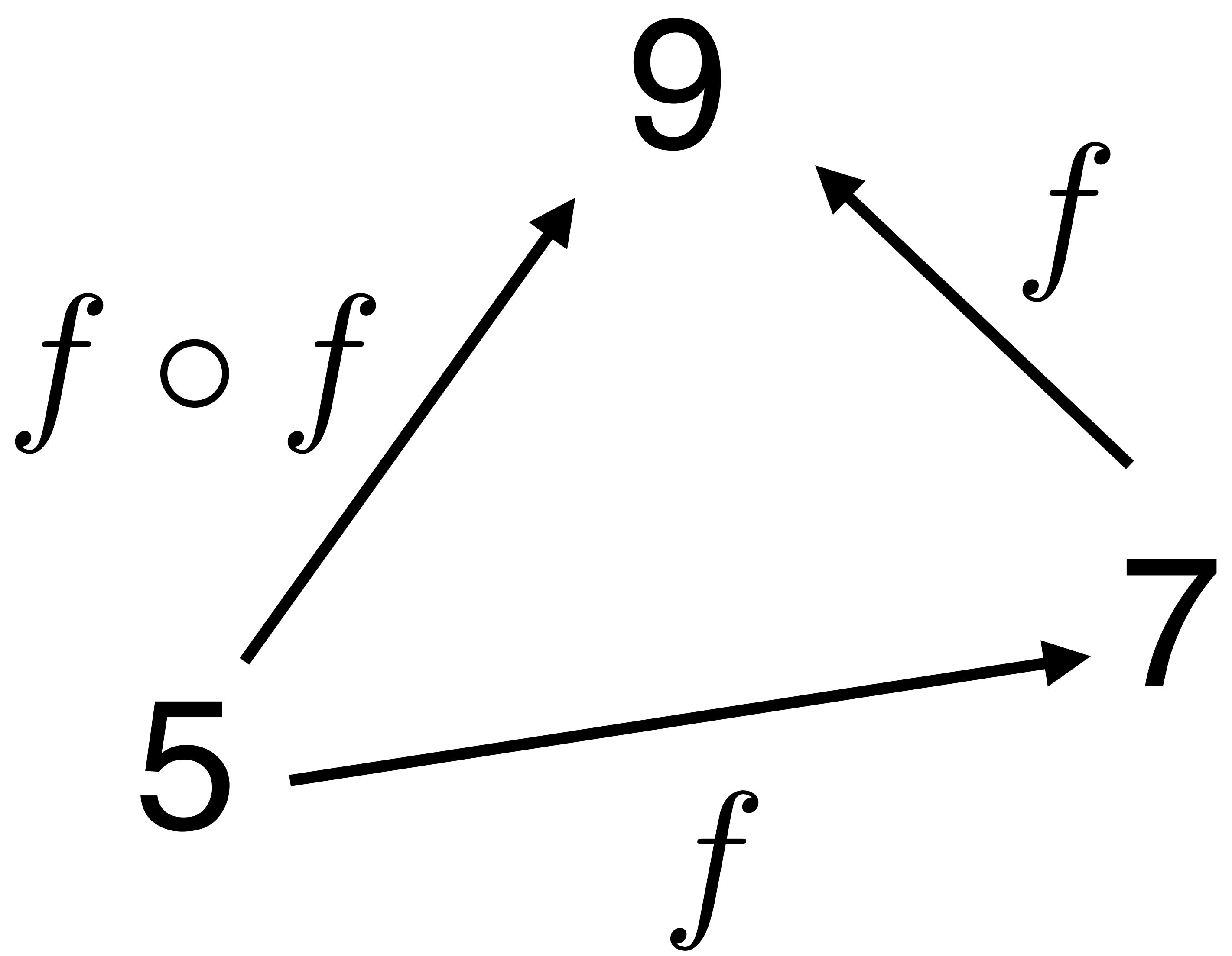
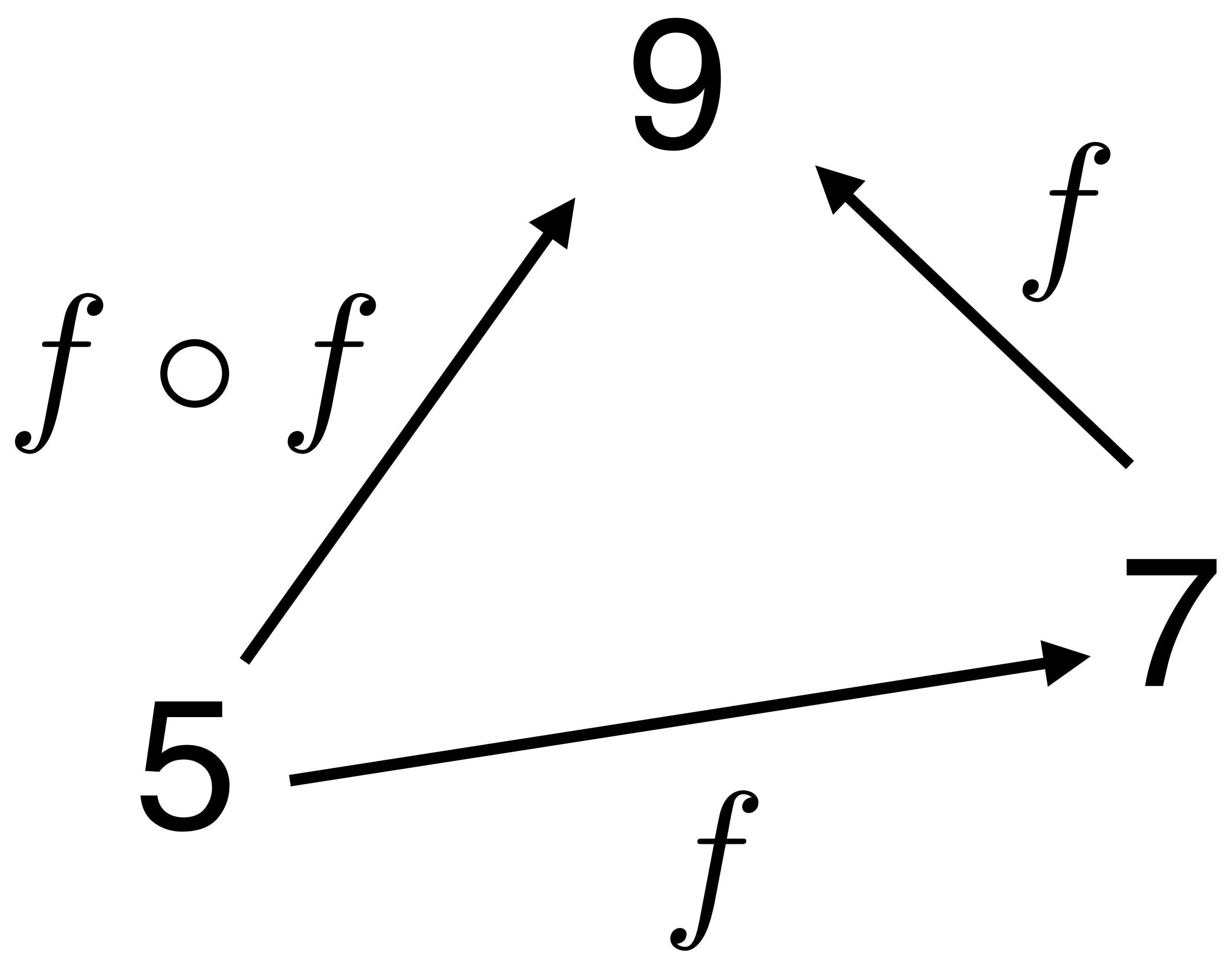


$$\forall a, b, c \in X : (aRb \land bRc) \Rightarrow aRc$$

But Hold On!



$$\forall a, b, c \in X : (aRb \wedge bRc) \Rightarrow aRc$$

But

$$f \circ f \neq f$$

Transitivity

$$\forall a, b, c \in X : (aRb \land bRc) \Rightarrow aRc$$

Transitivity

Composability

Transitivity

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$$F = GH \equiv aGbHc \Leftrightarrow aFc$$

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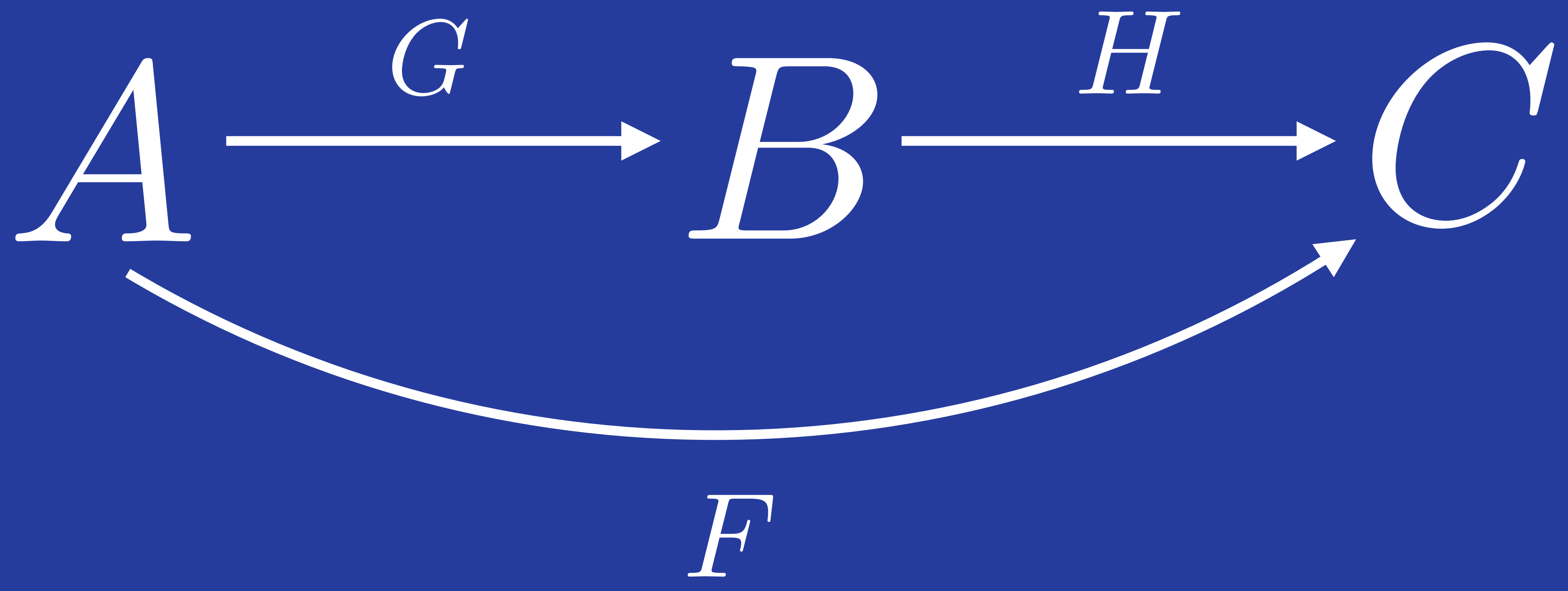


Transitivity

Composability

Transitivity

Composability

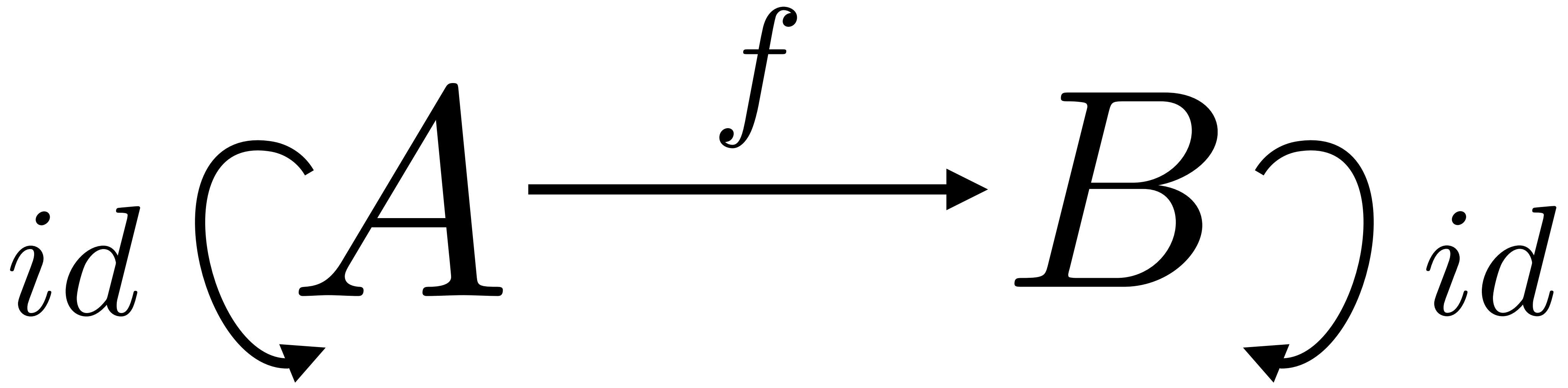
Composability

$$\circlearrowleft A \xrightarrow{f} B \circlearrowright$$

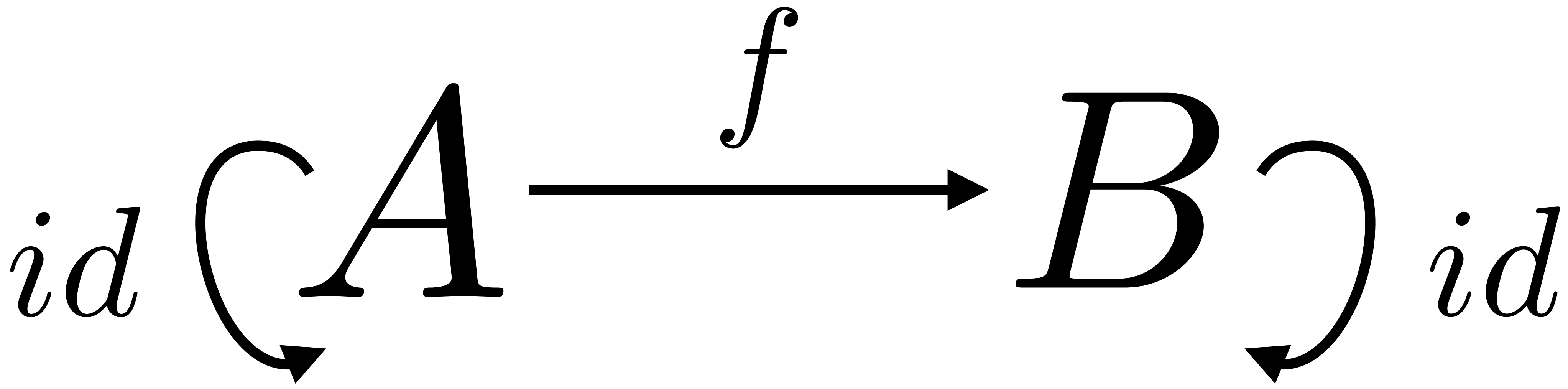
Composability

$$id \circlearrowleft A \xrightarrow{f} B \circlearrowright$$

Composability



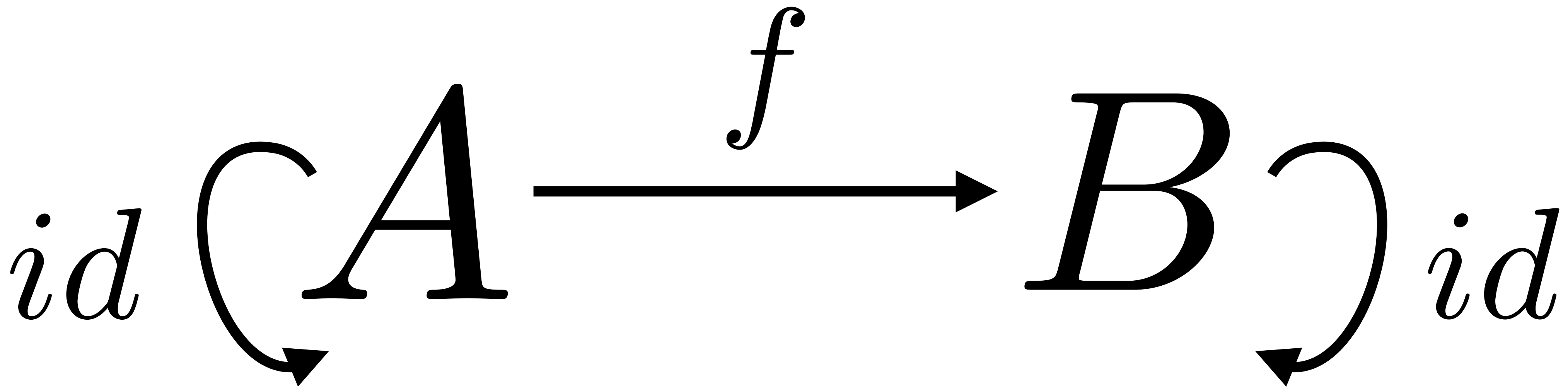
Composability



$$f \circ id = f$$

$$id \circ f = f$$

Identity



$$f \circ id = f$$

$$id \circ f = f$$

$$A \xrightarrow{f} B$$

Category

Category

A graph of arrows and objects.

Composability

$$F = GH \equiv aGbHc \Leftrightarrow aFc$$

Identity

$$f \circ id = f$$

$$id \circ f = f$$

Category

A collection (or class) of objects that have morphisms between them.

Composability

$$F = GH \equiv aGbHc \Leftrightarrow aFc$$

Identity

$$f \circ id = f$$

$$id \circ f = f$$

$$a \xrightarrow{f} b$$

Category

Morphism

Category

A collection (or class) of objects that have morphisms between them, which preserves composability and identity.

Set —functions→ Set

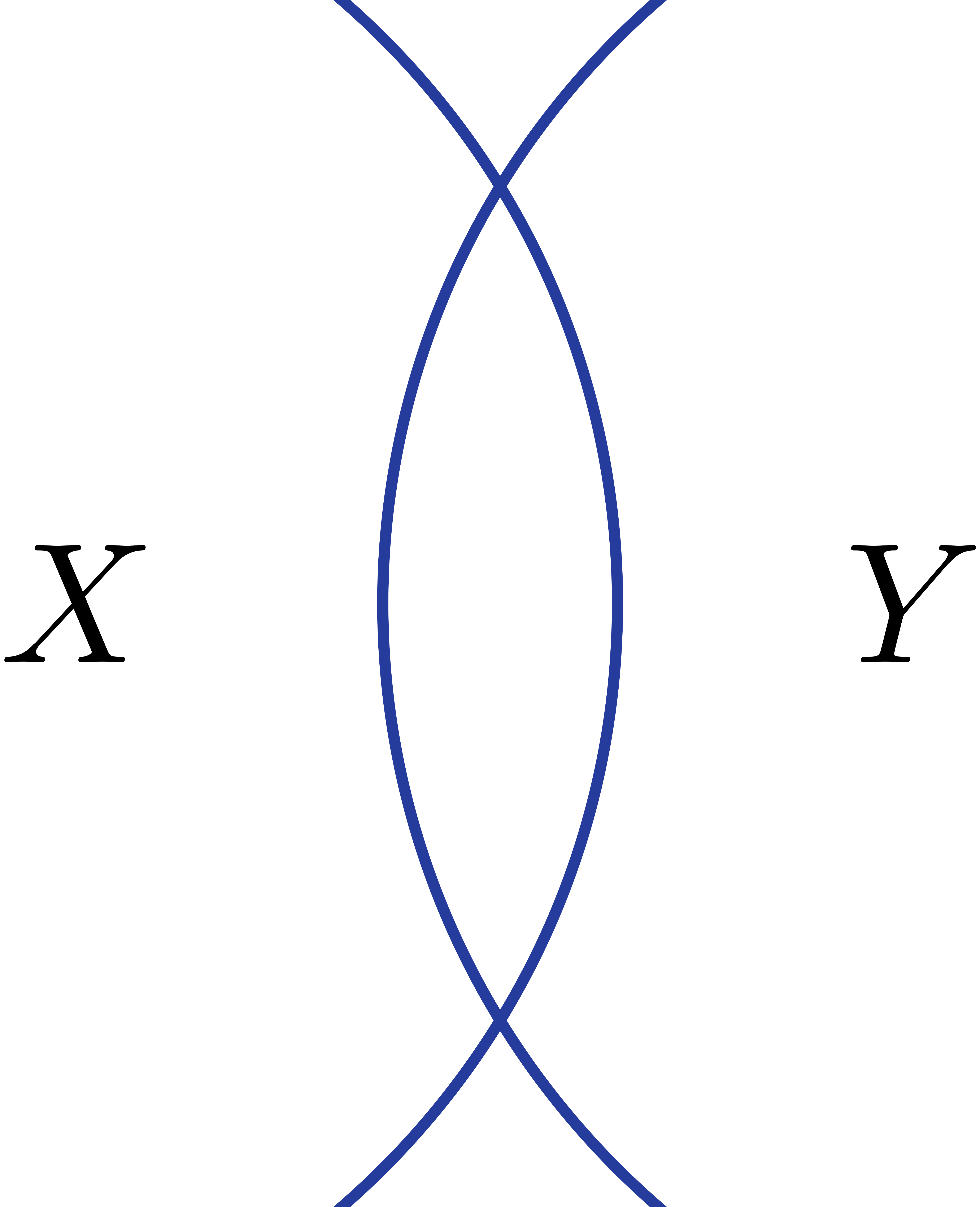
Group —group homomorphism→ Group

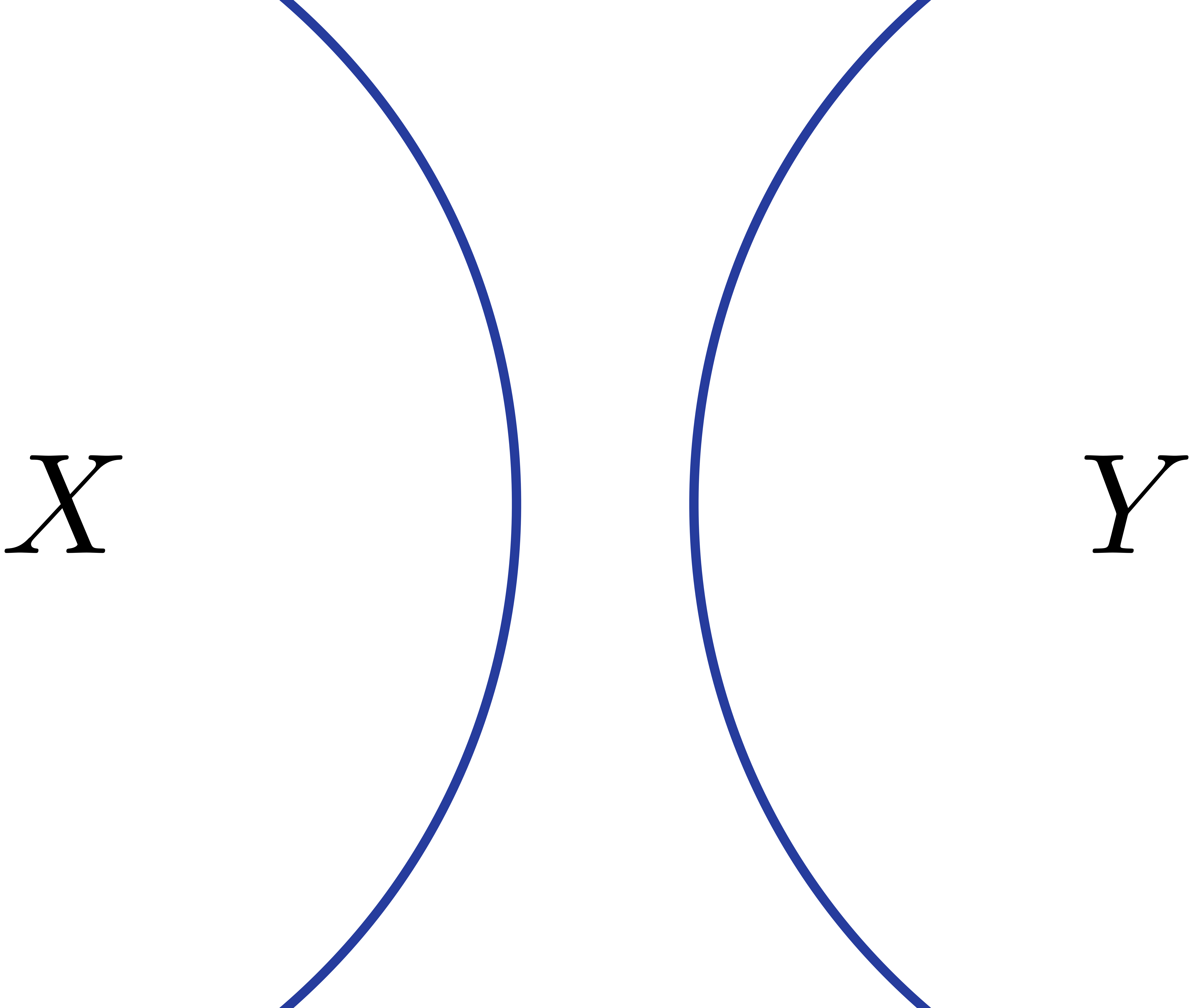
Vector Space —linear transformation→ Vector Space

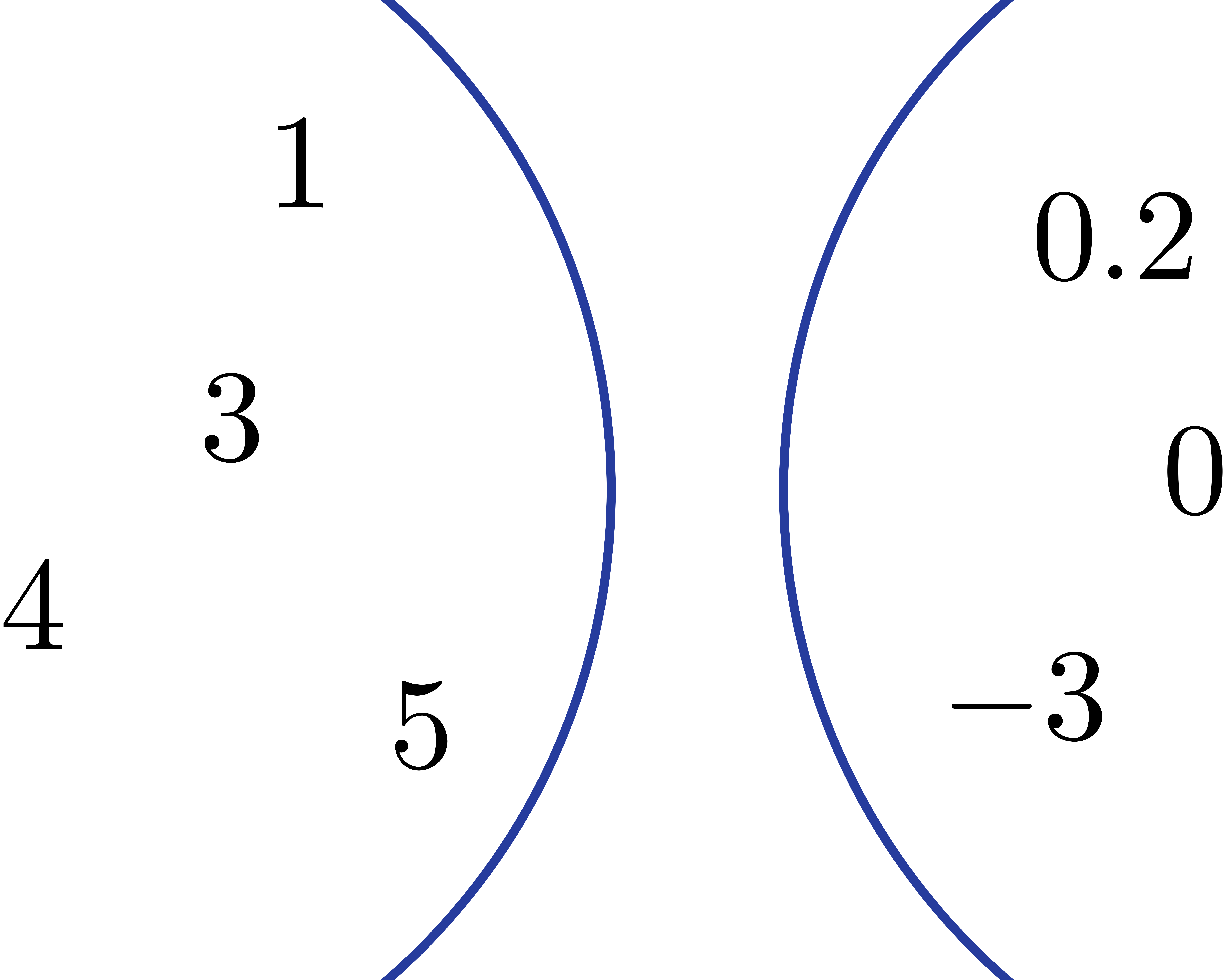
$$X \longrightarrow Y$$

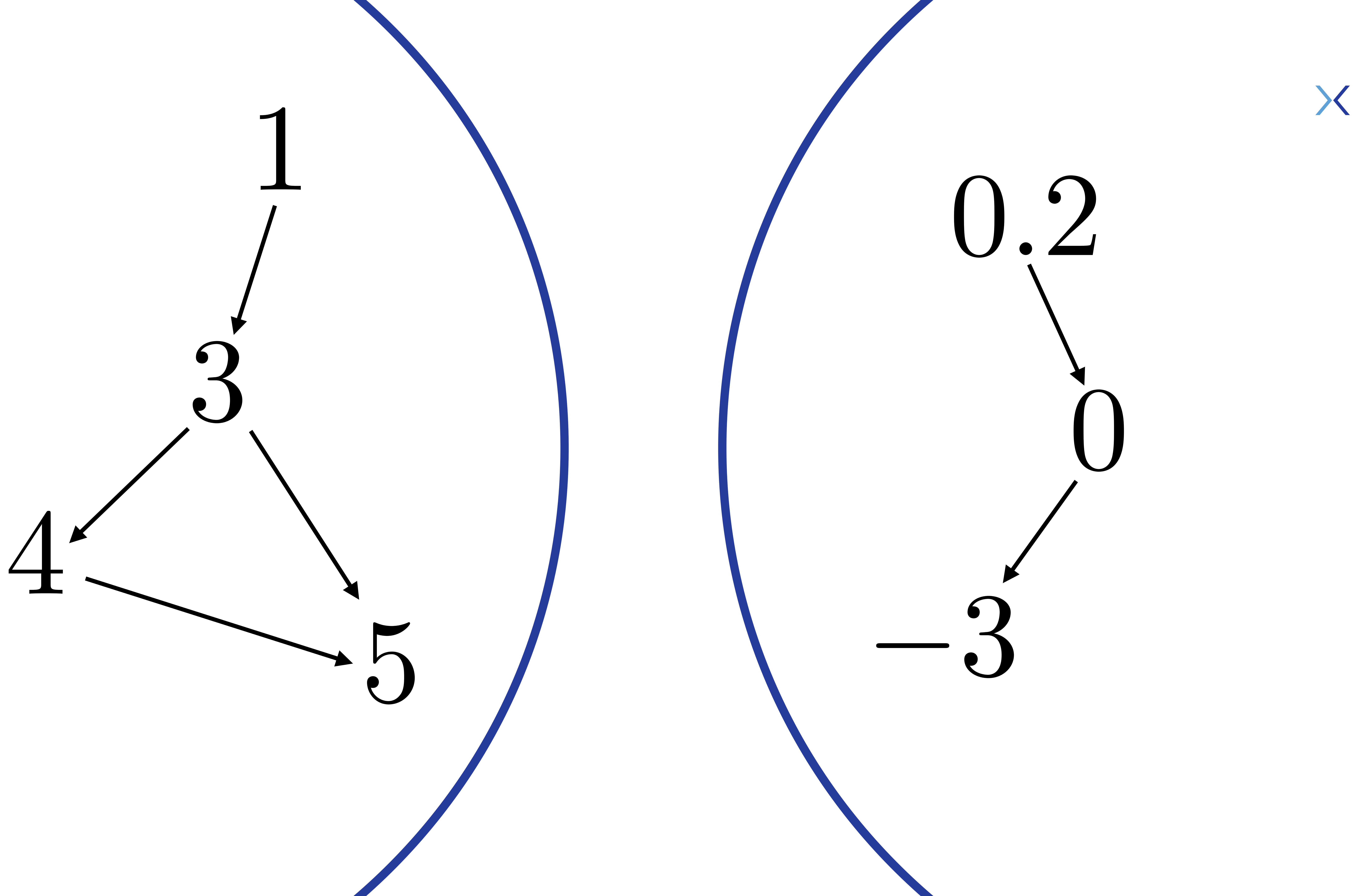
Functor

A morphism across categories

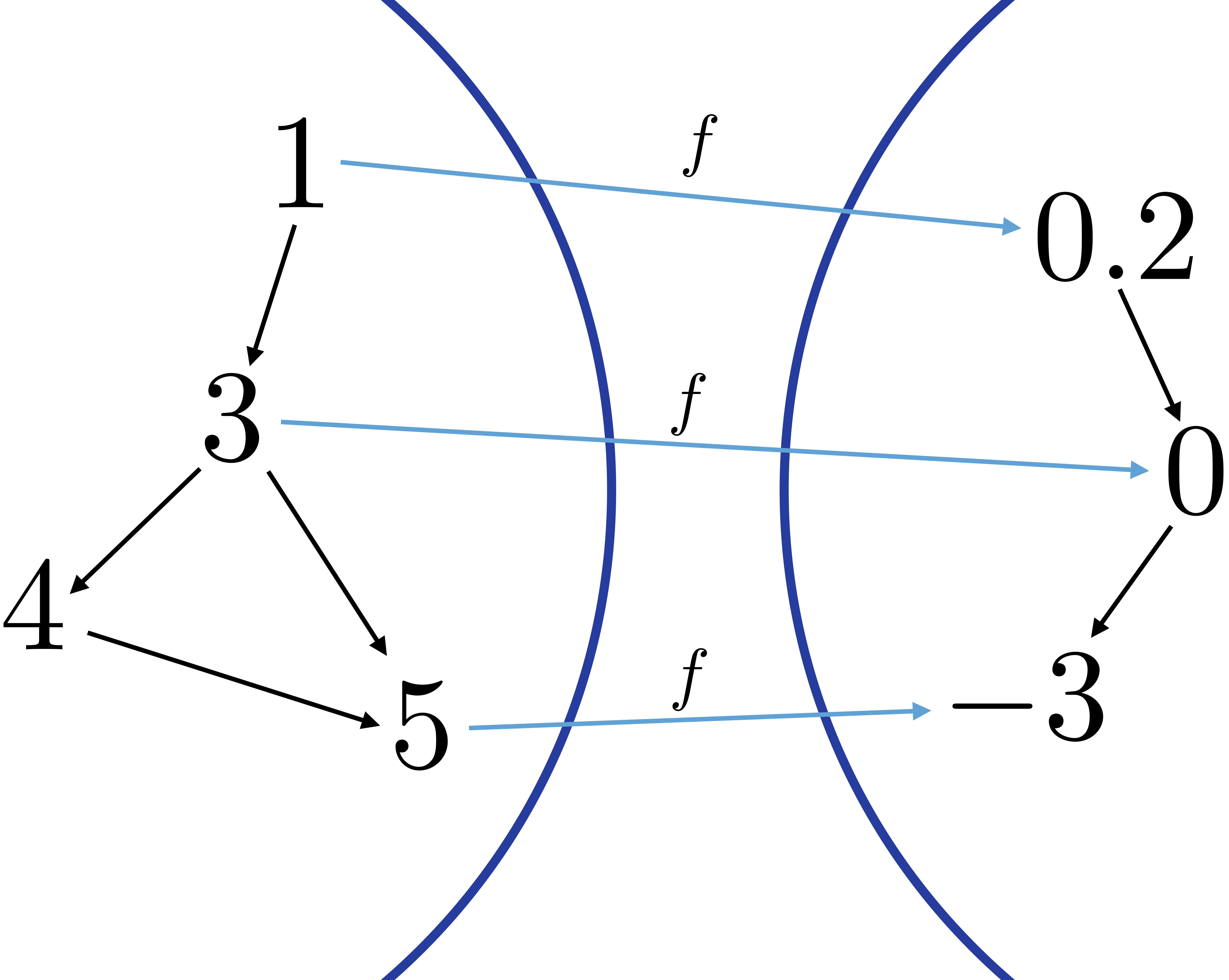

X
Y

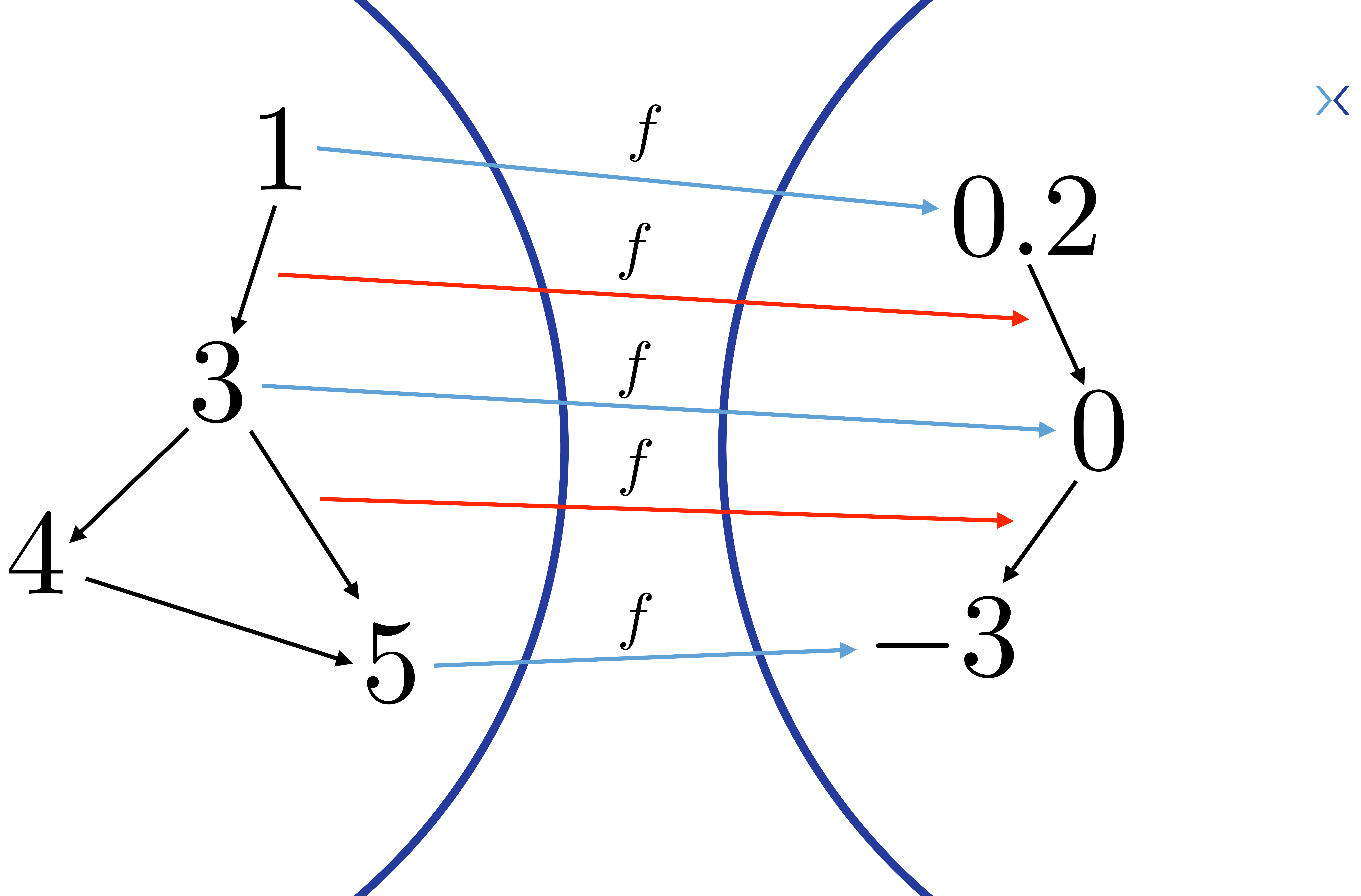

X
Y

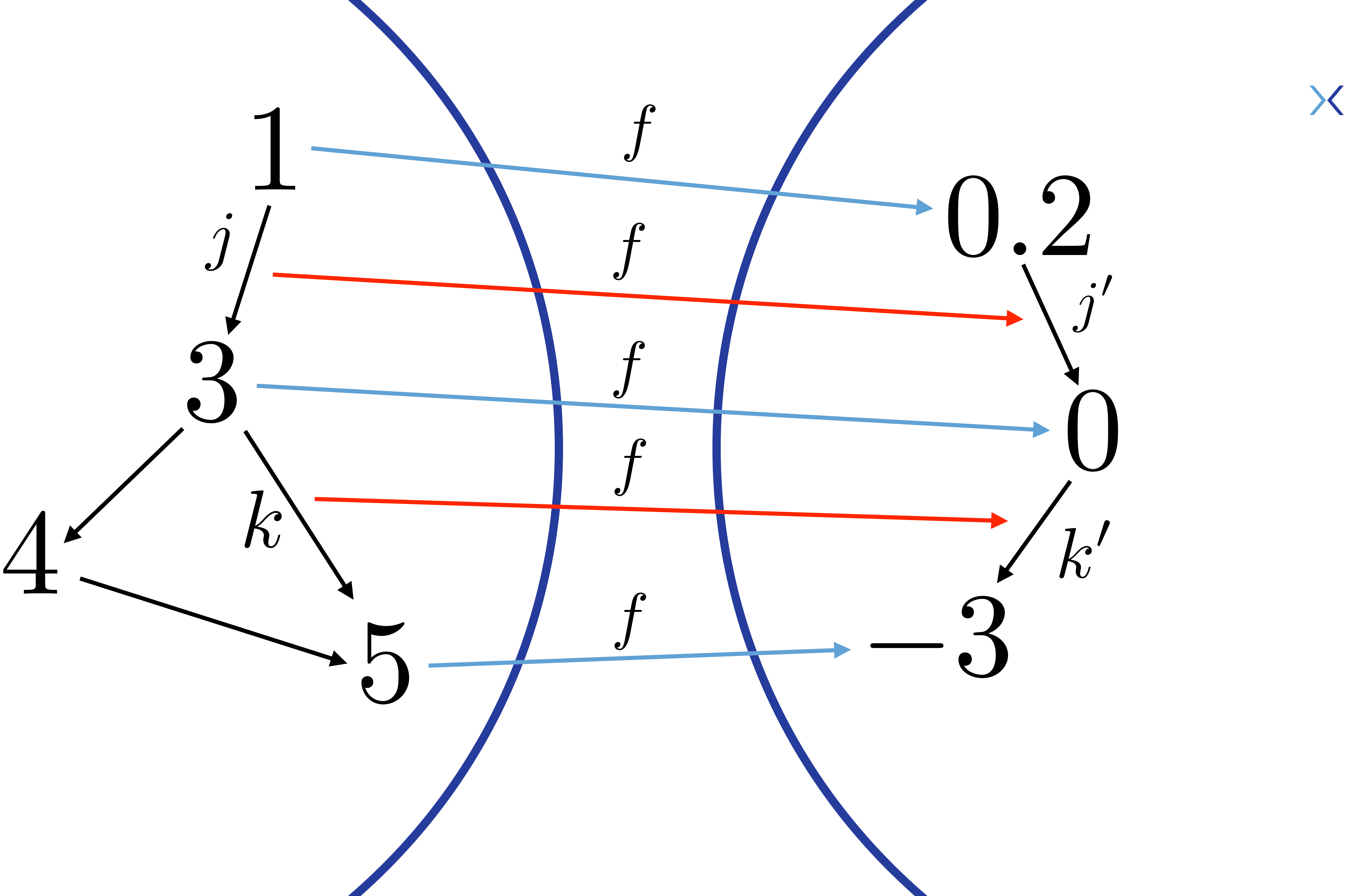

1
3
4
5
0.2
0
−3

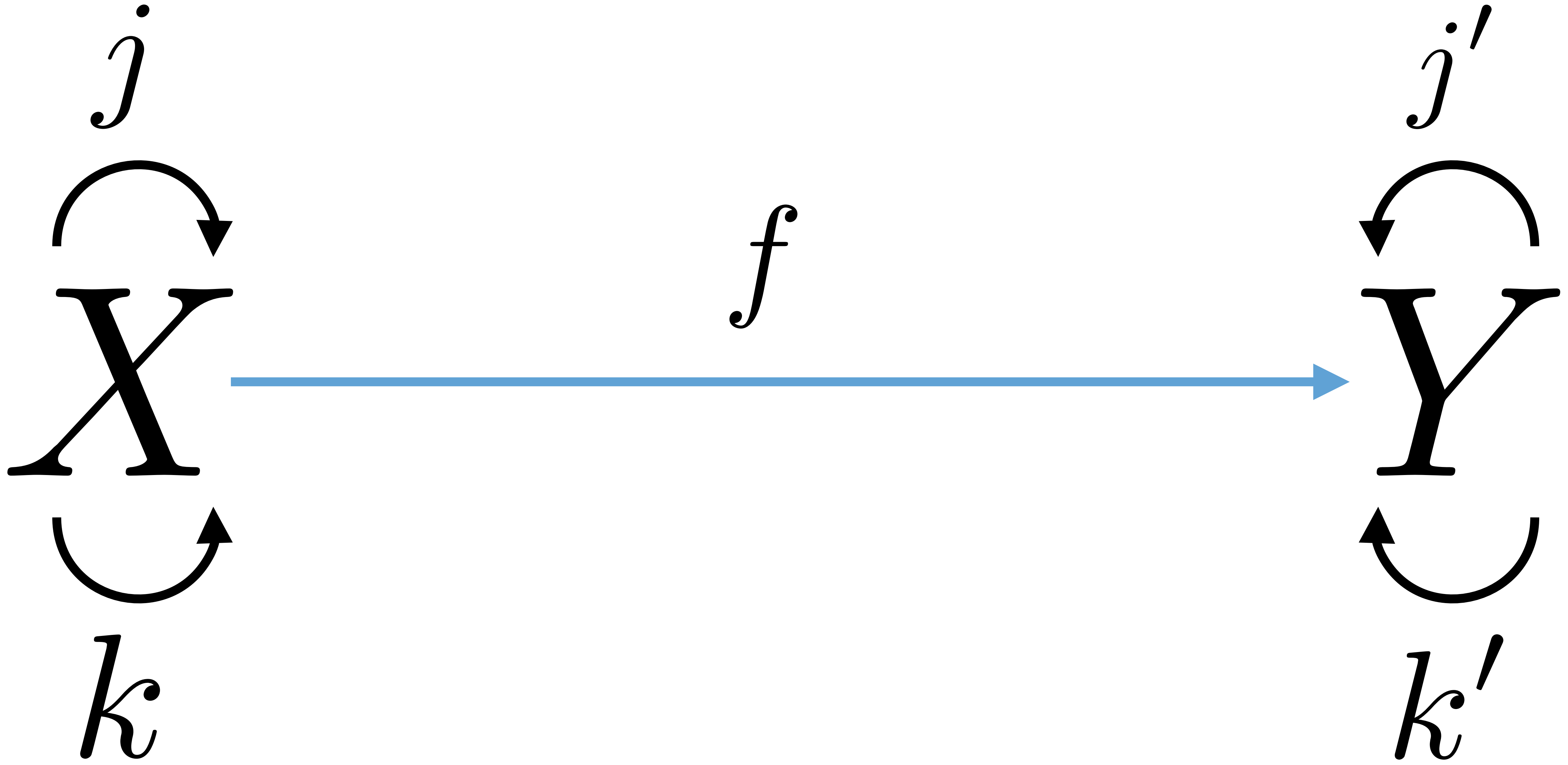


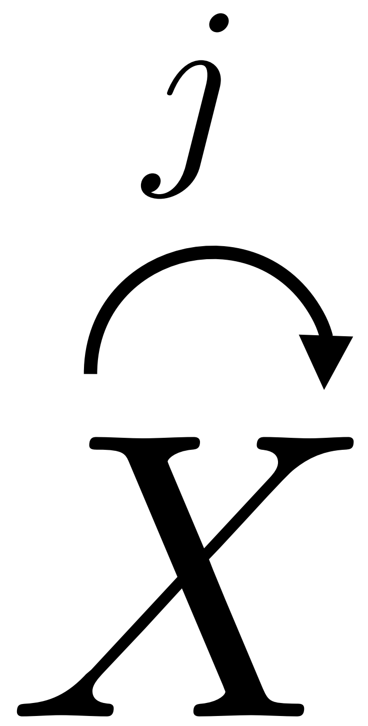
1
3
4
5
0.2
0
−3


1
3
4
5
f
f
f
0.2
0
−3


1
3
4
5
f
f
f
f
f
0.2
0
−3

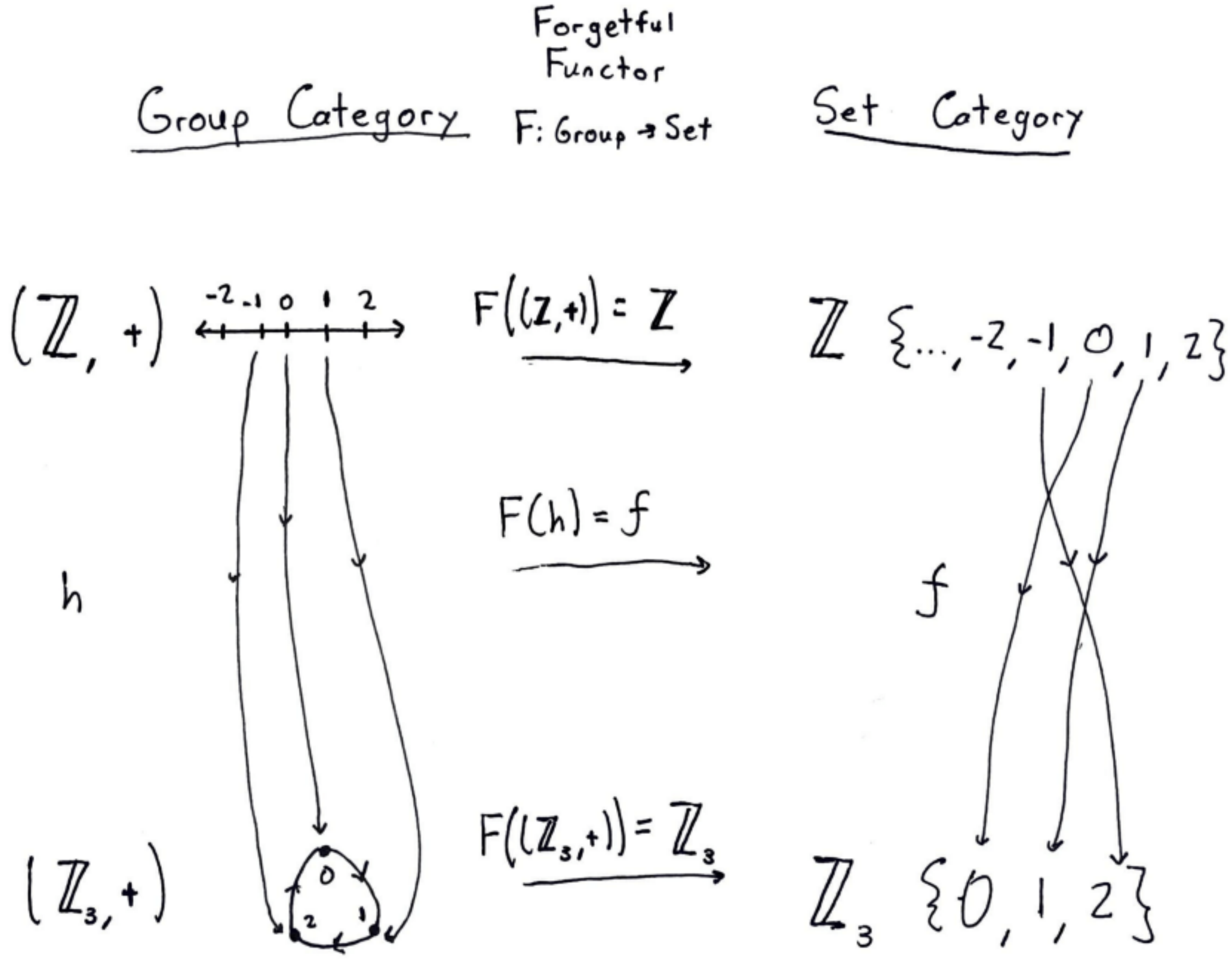

1
3
4
5
j
k
0.2
0
−3
j′
k′
f
f
f
f
f

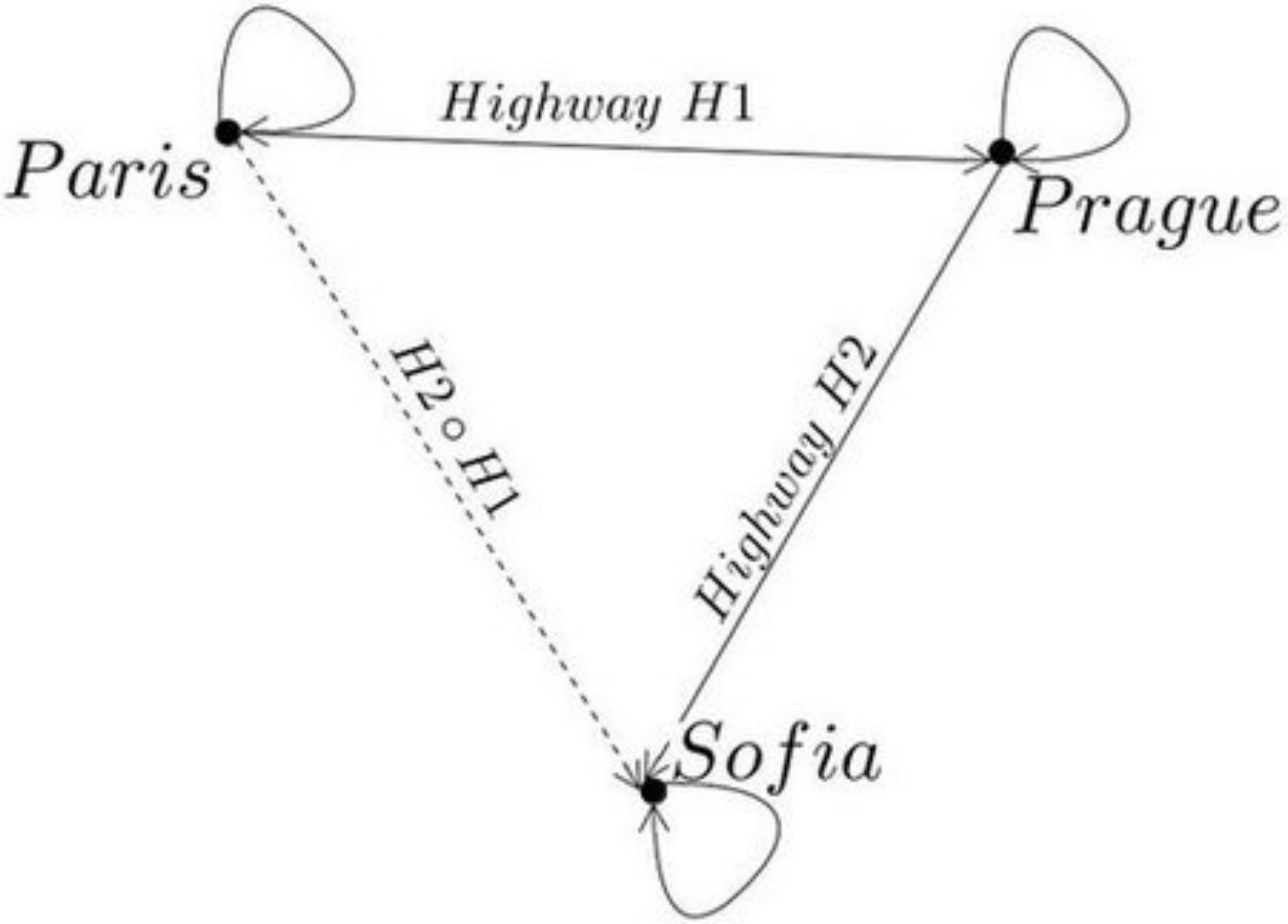

j
j'
f
X
Y
k
k'

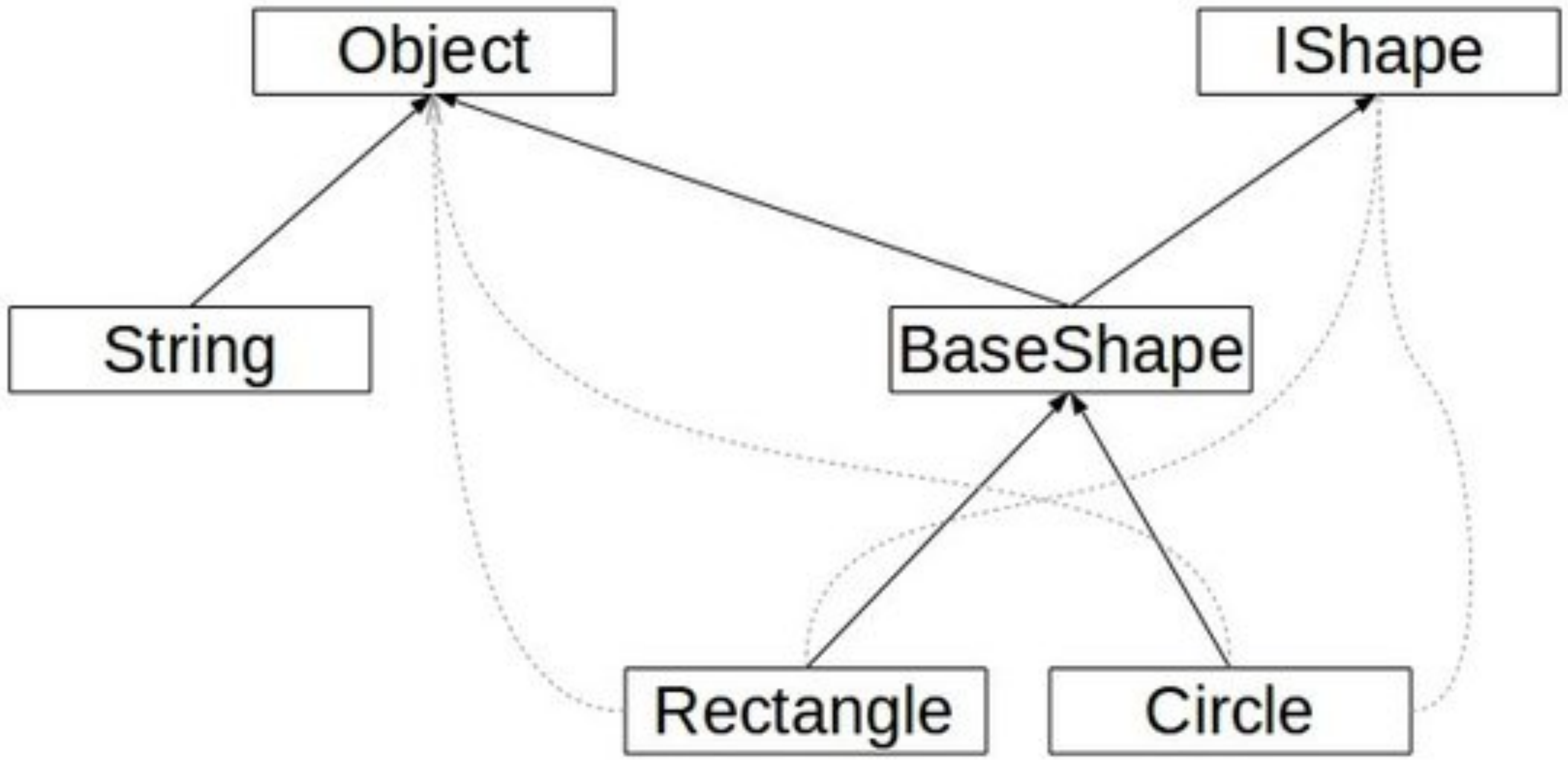


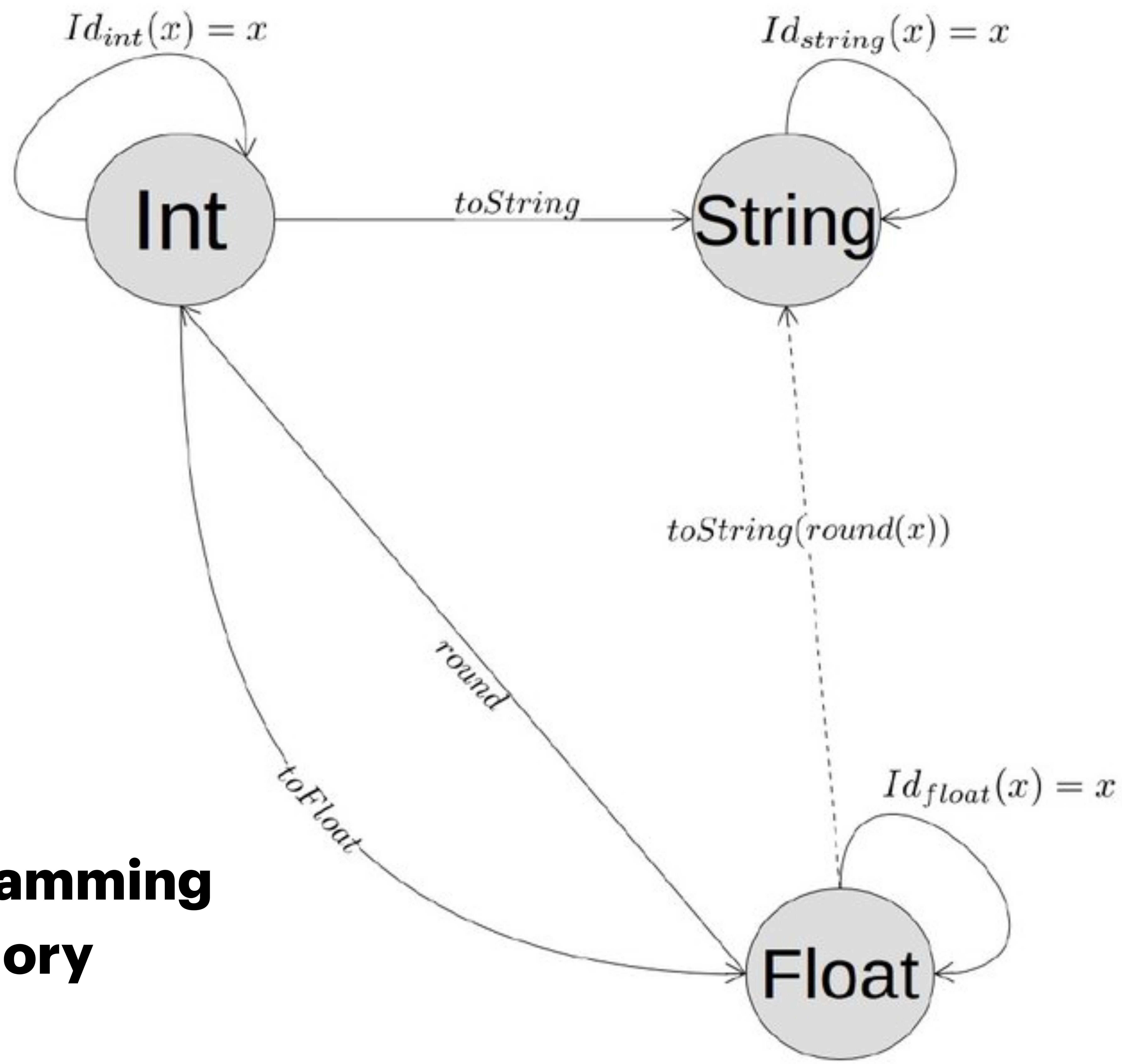
Endofunctor

A functor starting and ending in the same category


Forgetful Functor
Group Category
F: Group → Set
Set Category
(ℤ, +)
-2 -1 0 1 2
F((ℤ,+)) = ℤ
ℤ {..., -2, -1, 0, 1, 2}
h
F(h) = f
f
(ℤ3, +)
0
1
2
F((ℤ3,+)) = ℤ3
ℤ3 {0, 1, 2}


Highway H1
Paris
Prague
H2 ∘ H1
Highway H2
Sofia


Object
IShape
String
BaseShape
Rectangle
Circle



Functional Programming - Hask Category